

Global well-posedness and scattering for the energy-critical, defocusing Hartree equation in \mathbb{R}^{1+n}

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Abstract

Using the same induction on energy argument in both frequency space and spatial space simultaneously as in [6], [31] and [35], we obtain global well-posedness and scattering of energy solutions of defocusing energy-critical nonlinear Hartree equation in $\mathbb{R} \times \mathbb{R}^n (n \geq 5)$, which removes the radial assumption on the data in [25]. The new ingredients are that we use a modified long time perturbation theory to obtain the frequency localization (Proposition 3.1 and Corollary 3.1) of the minimal energy blow up solutions, which can not be obtained from the classical long time perturbation and bilinear estimate and that we obtain the spatial concentration of minimal energy blow up solution after proving that $L_x^{\frac{2n}{n-2}}$ -norm of minimal energy blow up solutions is bounded from below, the $L_x^{\frac{2n}{n-2}}$ -norm is larger than the potential energy.

Key Words: Hartree equation; Global well-posedness; Scattering; Minimal energy blow-up solutions; Frequency-localized interaction Morawetz estimate.

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1 Introduction

In this paper, we consider the following initial value problem

$$\begin{cases} iu_t + \Delta u = f(u), & \text{in } \mathbb{R}^n \times \mathbb{R}, \quad n \geq 5, \\ u(0) = u_0(x), & \text{in } \mathbb{R}^n. \end{cases} \quad (1.1)$$

where $u(t, x)$ is a complex-valued function in spacetime $\mathbb{R} \times \mathbb{R}^n$ and Δ is the Laplacian in \mathbb{R}^n , $f(u) = (|x|^{-4} * |u|^2)u = (|\nabla|^{-(n-4)}|u|^2)u$. It is introduced as a classical model in [37]. In practice, we use the integral formulation of (1.1)

$$u(t) = U(t)u_0(x) - i \int_0^t U(t-s)f(u(s))ds, \quad (1.2)$$

where $U(t) = e^{it\Delta}$.

This equation has the Hamiltonian

$$E(u(t)) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{4} \iint \frac{1}{|x-y|^4} |u(t, x)|^2 |u(t, y)|^2 dx dy. \quad (1.3)$$

Since (1.3) is preserved by the flow corresponding to (1.1) we shall refer to it as the energy and often write $E(u)$ for $E(u(t))$.

We are primarily interested in (1.1) since it is critical with respect to the energy norm. That is, the scaling $u \mapsto u_\lambda$ where

$$u_\lambda(t, x) = \lambda^{\frac{n-2}{2}} u(\lambda^2 t, \lambda x), \quad \lambda > 0 \quad (1.4)$$

leaves the energy invariant, in other words, the energy $E(u)$ is a dimensionless quantity.

As is well-known that if the initial data $u_0(x)$ has finite energy, then (1.1) is locally well-posed (see, for instance [3], [24]). That is, there exists a unique local-in-time solution that lies in $C_t^0 \dot{H}_x^1 \cap L_t^6 L_x^{\frac{6n}{3n-8}}$ and the map from the initial data to the solution is locally Lipschitz in these norms. If the energy is small, it is known that the solution exists globally in time and scattering occurs; However, for initial data with large energy, the local well-posedness argument do not extend to give global well-posedness, only with the conservation of the energy (1.3), because the lifespan of existence given by the local theory depends on the profile of the data as well as on the energy.

A large amount of works have been devoted to the theory of scattering for Hartree equation, see [4], [8]-[14], [22], [23] and [25]-[30]. In particular, global well-posedness and scattering of (1.1) with radial data in \dot{H}_x^1 was obtained in [25] by taking advantage of the term $-\int_I \int_{|x| \leq A|I|^{1/2}} |u|^2 \Delta \left(\frac{1}{|x|} \right) dx dt$ in the localized Morawetz identity to rule out the possibility of energy concentration at origin. In this paper, we continue this investigation. In order to prevent concentration at any location in spacetime, we should take advantage of the interaction Morawetz estimate achieved in [5], or the frequency-localized interaction Morawetz estimate in [6], see also [31], [35], [36], etc.

Here, we give their brief differences in the case of the defocusing Schrödinger equation. After proving the negative regularity of soliton solutions and double low-to-high frequency cascade solutions (some kinds of minimal energy blow up solutions or almost periodic solutions modulo symmetries), we can utilize the interaction Morawetz estimate to prevent the concentration of them at any location. While the negative regularity of soliton solutions and double low-to-high frequency cascade solutions can be obtained under the additional assumption of spatial dimension $n \geq 5$ due to the fact that the Schrödinger dispersion is not strong enough to perform the double Duhamel trick for the low dimensions $n = 3, 4$. But we can utilize the frequency localized interaction Morawetz estimate to prevent the concentration of them at any location in low dimensions as well as in high dimensions. See details in [6], [20], [31], [35], [36], etc.

Together with the frequency-localized interaction Morawetz estimate, we will use the same induction on energy argument in both frequency space and spatial space simultaneously as in [6] to obtain global well-posedness and scattering for general large data, which removes the radial assumption in [25]. As for induction on energy argument, we can also refer to [1], [31], [35] and [36]. Induction on energy argument is quantitative. In contrast with this method, D. Li, C. Miao and X. Zhang [22] recently use concentration compactness principle to obtain the similar result, that method is qualitative and firstly introduced by Kenig and Merle [16] to deal with the global well-posedness and scattering for focusing energy-critical NLS. There are many applications in this direction, for example [17], [19], [20], [21], etc.

However, the stability theory for the equation (1.1) is an essential tool for induction on energy argument. In the frame work of the classical long time perturbation, we inevitably demand to control the non-local interaction between the low and high frequencies

$$\left\| (|\nabla|^{-(n-4)} |u_{lo}|^2) u_{hi} \right\|_{L_t^{\frac{3}{2}} \dot{H}_x^1, \frac{6n}{3n+4}}(I \times \mathbb{R}^n)$$

with

$$\|u_{lo}\|_{\dot{S}^{1+s}(I \times \mathbb{R}^n)} \leq C(\eta) \epsilon^s, \quad \|u_{hi}\|_{\dot{S}^{1-k}(I \times \mathbb{R}^n)} \leq C(\eta) \epsilon^k, \quad \forall 0 \leq s, k \leq 1. \quad (1.5)$$

where the definition of norm $\|\cdot\|_{\dot{S}^1}$ refers to (1.7). Because $|\nabla|^{-(n-4)}$ destroys the direct interaction between u_{lo} and u_{hi} and we cannot use bilinear estimate to obtain any decay even though we have the

estimate (1.5). It is different from the local interaction case of the Schrödinger equation [6], [31] and [35]. We only have

$$\|(|\nabla|^{-(n-4)}|u_{lo}|^2)u_{hi}\|_{L_t^{\frac{3}{2}}\dot{H}_x^{1, \frac{6n}{3n+4}}(I \times \mathbb{R}^n)} \leq C(\eta).$$

No decay dues to one derivative on spatial variable in the spacetime space $L_t^{\frac{3}{2}}\dot{H}_x^{1, \frac{6n}{3n+4}}(I \times \mathbb{R}^n)$. In deed, because we can not use the bilinear estimate when one derivative falls on u_{hi} , there is no any decay by (1.5). However, when we would like to transfer some part of derivative to integral on spatial variable, we can obtain the small interaction

$$\|(|\nabla|^{-(n-4)}|u_{lo}|^2)u_{hi}\|_{L_t^{\frac{3}{2}}(I; \dot{H}_x^{1-\frac{2}{n}, \frac{6n^2}{3n^2+4n-12}})} \leq C(\eta)\epsilon^{\frac{4}{n}}$$

according to (1.5). Inspired by this fact together with the inhomogeneous Strichartz estimate [7], [15] and [34], we set down a modified long time perturbation, which replaces the role of the classical long time perturbation and the bilinear estimate in some senses and is important to establish the frequency localization (Proposition 3.1 and Corollary 3.1) of minimal energy blow up solutions. See details in Section 4.

In addition, we obtain the spatial concentration of minimal energy blow up solution after we prove that $L_x^{\frac{2n}{n-2}}$ -norm of minimal energy blow up solutions is bounded from below, which is stronger than the statement that the potential energy of minimal energy blow up solution is bounded from below.

Now, we give the main result of this paper.

Theorem 1.1. *Let $n \geq 5$. For any u_0 with finite energy, $E(u_0) < \infty$, there exists a unique global solution $u \in C_t^0(\dot{H}_x^1) \cap L_t^6(L_x^{\frac{6n}{3n-8}})$ to (1.1) such that*

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(\mathbb{R} \times \mathbb{R}^n)} \leq C(E(u_0)) \quad (1.6)$$

for some constant $C(E(u_0))$ that depends only on the energy.

As is well-known, the $L_t^6 L_x^{\frac{6n}{3n-8}}$ bound above also gives scattering, asymptotic completeness, and uniform regularity.

Corollary 1.1. *Let u_0 have finite energy. Then there exist finite energy solutions $u_{\pm}(t, x)$ to the free Schrödinger equation $(i\partial_t + \Delta)v = 0$ such that*

$$\|u(t) - u_{\pm}(t)\|_{\dot{H}^1} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty.$$

Furthermore, the maps $u_0 \mapsto u_{\pm}(0)$ are homeomorphisms from \dot{H}^1 to \dot{H}^1 . Finally, if $u_0 \in H^s$ for some $s > 1$, then $u(t) \in H^s$ for all time t , and one has the uniform bounds

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C(E(u_0), s) \|u_0\|_{H^s}.$$

Next, we introduce some notations. If X, Y are nonnegative quantities, we use $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $X \leq CY$ for some C which may depend on the critical energy E_{crit} (see Section 3) but not on any parameter such as η , and $X \sim Y$ to denote the estimate $X \lesssim Y \lesssim X$. We use $X \ll Y$ to mean $X \leq cY$ for some small constant c which is again allowed to depend on E_{crit} . We use $C \gg 1$ to denote various large finite constants, and $0 < c \ll 1$ to denote various small constants.

We use $L_t^q L_x^r$ to denote the spacetime norm

$$\|u\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t, x)|^r dx \right)^{q/r} dt \right)^{1/q}$$

with the usual modifications when q or r is infinity, or when the domain $\mathbb{R} \times \mathbb{R}^n$ is replaced by some smaller spacetime region. When $q = r$, we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$.

When $n \geq 5$, we say that a pair (q, r) is sharp admissible if

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right), \quad 2 \leq r \leq \frac{2n}{n-2}.$$

We say that the pair (q, r) is acceptable if

$$1 \leq q, r \leq \infty, \quad \frac{1}{q} < n\left(\frac{1}{2} - \frac{1}{r}\right), \quad \text{or} \quad (q, r) = (\infty, 2).$$

For a spacetime slab $I \times \mathbb{R}^n$, we define the *Strichartz* norm $\dot{S}^0(I)$ by

$$\|u\|_{\dot{S}^0(I)} := \sup_{(q,r) \text{ sharp admissible}} \left(\sum_N \|P_N u\|_{L_t^q L_x^r(I \times \mathbb{R}^n)}^2 \right)^{1/2}.$$

and for $k > 0$, we define $\dot{S}^k(I)$ by

$$\|u\|_{\dot{S}^k(I)} := \| |\nabla|^k u \|_{\dot{S}^0(I)}. \quad (1.7)$$

From the Littlewood-Paley inequality, Sobolev embedding and Minkowski's inequality, we have

$$\begin{aligned} & \|\nabla u\|_{L_t^\infty L_x^2} + \|\nabla u\|_{L_t^6 L_x^{\frac{6n}{3n-2}}} + \|\nabla u\|_{L_t^{\frac{6(n-2)}{n}} L_x^{\frac{6(n-2)}{3n-8}}} + \|\nabla u\|_{L_t^3 L_x^{\frac{6n}{3n-4}}} + \|\nabla u\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \\ & + \|u\|_{L_t^\infty L_x^{\frac{2n}{n-2}}} + \|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}} + \|u\|_{L_t^4 L_x^{\frac{2n}{n-3}}} + \|u\|_{L^3 L^{\frac{6n}{3n-10}}} + \|u\|_{L_t^2 L_x^{\frac{2n}{n-4}}} \lesssim \|u\|_{\dot{S}^1}, \end{aligned} \quad (1.8)$$

where all spacetime norms are taken on $I \times \mathbb{R}^n$.

The Fourier transform on \mathbb{R}^n is defined by

$$\widehat{f}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

giving rise to the fractional differentiation operators $|\nabla|^s$, defined by

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \widehat{f}(\xi).$$

These define the homogeneous Sobolev norms

$$\|f\|_{\dot{H}_x^s} := \| |\nabla|^s f \|_{L_x^2(\mathbb{R}^n)}.$$

Let $e^{it\Delta}$ be the free Schrödinger propagator. This propagator preserves the above Sobolev norms and obeys the dispersive estimate

$$\|e^{it\Delta} f\|_{L_x^\infty(\mathbb{R}^n)} \lesssim |t|^{-\frac{n}{2}} \|f\|_{L_x^1(\mathbb{R}^n)} \quad (1.9)$$

for all times $t \neq 0$. We also recall *Duhamel's* formula

$$u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-s)\Delta} f(u)(s) ds. \quad (1.10)$$

We will occasionally use subscripts to denote spatial derivatives and will use the summation convention over repeated indices.

We will also need the Littlewood-Paley projection operators. Specifically, let $\varphi(\xi)$ be a smooth bump function adapted to the ball $|\xi| \leq 2$ which equals 1 on the ball $|\xi| \leq 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$, we define the Littlewood-Paley operators

$$\begin{aligned}\widehat{P_{\leq N}f}(\xi) &:= \varphi\left(\frac{\xi}{N}\right)\widehat{f}(\xi), & \widehat{P_{>N}f}(\xi) &:= \left(1 - \varphi\left(\frac{\xi}{N}\right)\right)\widehat{f}(\xi), \\ \widehat{P_Nf}(\xi) &:= \left(\varphi\left(\frac{\xi}{N}\right) - \varphi\left(\frac{2\xi}{N}\right)\right)\widehat{f}(\xi).\end{aligned}$$

Similarly we can define $P_{<N}$, $P_{\geq N}$, and $P_{M < \cdot \leq N} = P_{\leq N} - P_{\leq M}$, whenever M and N are dyadic numbers. We will frequently write $f_{\leq N}$ for $P_{\leq N}f$ and similarly for the other operators.

The Littlewood-Paley operators commute with derivative operators, the free propagator, and the conjugation operation. They are self-adjoint and bounded on every L_x^p and \dot{H}_x^s space for $1 \leq p \leq \infty$ and $s \geq 0$. They also obey the following Sobolev and Bernstein estimates

$$\begin{aligned}\|P_{\geq N}f\|_{L^p} &\lesssim N^{-s} \|\nabla^s P_{\geq N}f\|_{L^p}, \\ \|\nabla^s P_{\leq N}f\|_{L^p} &\lesssim N^s \|P_{\leq N}f\|_{L^p}, & \|P_{\leq N}f\|_{L^q} &\lesssim N^{\frac{n}{p} - \frac{n}{q}} \|P_{\leq N}f\|_{L^p}, \\ \|\nabla^{\pm s} P_N f\|_{L^p} &\sim N^{\pm s} \|P_N f\|_{L^p}, & \|P_N f\|_{L^q} &\lesssim N^{\frac{n}{p} - \frac{n}{q}} \|P_N f\|_{L^p},\end{aligned}$$

whenever $s \geq 0$ and $1 \leq p \leq q \leq \infty$. Note that the kernel of the operator $P_{\leq N}$ is not positive. To overcome this problem, we use the operator $P'_{\leq N}$ in [35], etc. More precisely, if $K_{\leq N}$ is the kernel associated to $P_{\leq N}$, we let $P'_{\leq N}$ be the operator associated to $N^{-n}(K_{\leq N})^2$. The kernel of $P'_{\leq N}$ is bounded in L_x^1 independently of N . Therefore, the operator $P'_{\leq N}$ is bounded on every L_x^p for $1 \leq p \leq \infty$. Furthermore, for $s \geq 0$ and $1 \leq p \leq q \leq \infty$, we have

$$\|\nabla^s P'_{\leq N}f\|_{L_x^p} \lesssim N^s \|P'_{\leq N}f\|_{L_x^p}, \quad \|P'_{\leq N}f\|_{L_x^q} \lesssim N^{\frac{n}{q} - \frac{n}{p}} \|P'_{\leq N}f\|_{L_x^p}.$$

Last, the paper is organized as follows. In Section 2, we introduce Strichartz estimates and perturbation theory in \mathbb{R}^{1+n} ; In Section 3, we overview the proof of main theorem; In Section 4, we show that the frequency delocalization at one time implies spacetime bound, which means that the frequency localization of minimal energy blow up solutions; In Section 5, we show that $L_x^{2n/(n-2)}$ -norm of minimal energy blow up solutions is bounded from below, which means that the spatial concentration of minimal energy blow up solutions; In Section 6, we establish the frequency-localized interaction Morawetz estimate of minimal energy blow up solutions, which is used to eliminate soliton-like solutions; Finally, we prevent energy evacuation of minimal energy blow up solutions in Section 7, which is used to exclude the finite time blow-up solutions and double low-to-high frequency cascade solutions.

2 Strichartz estimates and perturbation theory in \mathbb{R}^{1+n}

In this section, we recall Strichartz estimates and the classical long time perturbation in $\mathbb{R} \times \mathbb{R}^n$ for $n \geq 5$, and give a modified long time perturbation, which is important to establish the frequency localization of minimal energy blow up solutions.

The following *Strichartz* inequalities are tied up with the local well-posedness theory.

Lemma 2.1 ([3], [6], [18], [32]). *Let I be a compact time interval, and Let $u : I \times \mathbb{R}^n \rightarrow \mathbb{C}$ be a Schwartz solution to the forced Schrödinger equation*

$$iu_t + \Delta u = \sum_{m=1}^M F_m$$

for some Schwartz functions F_1, \dots, F_M . Then

$$\|u\|_{\dot{S}^k(I \times \mathbb{R}^n)} \lesssim \|u(t_0)\|_{\dot{H}_x^k} + C \sum_{m=1}^M \|\nabla^k F_m(u)\|_{L_t^{q'_m} L_x^{r'_m}(I \times \mathbb{R}^n)}$$

for any $k \geq 0$ and $t_0 \in I$, and any sharp admissible pairs $(q_1, r_1), \dots, (q_M, r_M)$, where we use p' to denote the dual exponent to p , i.e. $1/p' + 1/p = 1$.

On the inhomogeneous Strichartz estimate, we also have

Lemma 2.2 (Inhomogenous Strichartz estimate, [7], [15], [34]). *If v is the solution of*

$$iv_t + \Delta v = F(t, x)$$

with zero data and inhomogeneous term F supported on $\mathbb{R} \times \mathbb{R}^d$, then we have the estimate

$$\|v\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

whenever (q, r) , (\tilde{q}, \tilde{r}) are acceptable, verify the scaling condition

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{n}{2} \left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right),$$

and either the conditions

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \quad \frac{n-2}{n} \leq \frac{r}{\tilde{r}} \leq \frac{n}{n-2},$$

or the conditions

$$\frac{1}{q} + \frac{1}{\tilde{q}} = 1, \quad \frac{n-2}{n} \leq \frac{r}{\tilde{r}} \leq \frac{n}{n-2}, \quad \frac{1}{r} \leq \frac{1}{q}, \quad \frac{1}{\tilde{r}} \leq \frac{1}{\tilde{q}}.$$

Now, similar as in [6], [31], [33] and [35], we first have

Lemma 2.3 (Classical long time perturbation). *Let I be a compact interval, and let \tilde{u} be a function on $I \times \mathbb{R}^n$ which obeys the bounds*

$$\|\tilde{u}\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I \times \mathbb{R}^n)} \leq M$$

and

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^n)} \leq E$$

for some $M, E > 0$. Suppose also that \tilde{u} is a near-solution to (1.1) in the sense that it solves

$$(i\partial_t + \Delta)\tilde{u} = (|x|^{-4} * |\tilde{u}|^2)\tilde{u} + e \tag{2.1}$$

for some function e . Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}_x^1(\mathbb{R}^n)} \leq E'$$

for some $E' > 0$. Assume also that we have the smallness conditions

$$\begin{aligned} & \left(\sum_N \|P_N \nabla e^{i(t-t_0)\Delta} (u(t_0) - \tilde{u}(t_0))\|_{L_t^6 L_x^{\frac{6n}{3n-2}}(I \times \mathbb{R}^n)}^2 \right)^{\frac{1}{2}} \\ & + \left(\sum_N \|P_N \nabla e^{i(t-t_0)\Delta} (u(t_0) - \tilde{u}(t_0))\|_{L_t^3 L_x^{\frac{6n}{3n-4}}(I \times \mathbb{R}^n)}^2 \right)^{\frac{1}{2}} \leq \epsilon, \end{aligned} \tag{2.2}$$

$$\|e\|_{L_t^{\frac{3}{2}} \dot{H}_x^{1, \frac{6n}{3n+4}}(I \times \mathbb{R}^n)} \leq \epsilon \tag{2.3}$$

for some $0 < \epsilon < \epsilon_1$, where ϵ_1 is some constant $\epsilon_1 = \epsilon_1(E, E', M) > 0$.

We conclude that there exists a solution u to (1.1) on $I \times \mathbb{R}^n$ with the specified initial data $u(t_0)$ at t_0 , and furthermore

$$\begin{aligned} \|u - \tilde{u}\|_{\dot{S}^1(I \times \mathbb{R}^n)} &\leq C(M, E, E'), \\ \|u\|_{\dot{S}^1(I \times \mathbb{R}^n)} &\leq C(M, E, E'), \\ \|u - \tilde{u}\|_{L_t^6 L_x^{\frac{6n}{3n-8}}} + \|u - \tilde{u}\|_{L_t^3 \dot{H}_x^{1, \frac{6n}{3n-4}}} &\leq C(M, E, E')\epsilon. \end{aligned}$$

Remark 2.1. Note that $u(t_0) - \tilde{u}(t_0)$ is allowed to have large energy, albeit at the cost of forcing ϵ to be smaller, and worsening the bounds in $\|u - \tilde{u}\|_{\dot{S}^1(I \times \mathbb{R}^n)}$. From the Strichartz estimate and Plancherel's theorem, we have

$$\begin{aligned} \text{L.H.S of (2.2)} &\lesssim \left(\sum_N \|P_N \nabla(u(t_0) - \tilde{u}(t_0))\|_{L_x^2}^2 \right)^{1/2} \\ &\lesssim \|\nabla(u(t_0) - \tilde{u}(t_0))\|_{L_x^2} \\ &\lesssim E'. \end{aligned}$$

Hence, the hypotheses (2.2) are redundant if one is willing to take $E' = O(\epsilon)$.

Based on Lemma 2.2, we can also obtain the following long time perturbation,

Lemma 2.4 (Modified long time perturbation). *Let I be a compact interval, and let \tilde{u} be a function on $I \times \mathbb{R}^d$ which obeys the bounds*

$$\|\tilde{u}\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I \times \mathbb{R}^n)} \leq M \quad (2.4)$$

and

$$\|\tilde{u}\|_{L_t^\infty(I; \dot{H}_x^1)} \leq E \quad (2.5)$$

for some $M, E > 0$. Suppose also that \tilde{u} is a near-solution to (1.1) in the sense that it solves (2.1) for some function e . Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}_x^1} \leq E'$$

for some $E' > 0$. Assume also that we have the smallness conditions

$$\|e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L_t^3(I; \dot{H}_x^{1-\frac{2}{n}, \frac{6n^2}{3n^2-4n-12}})} \leq \epsilon, \quad (2.6)$$

$$\|e\|_{L_t^{\frac{3}{2}}(I; \dot{H}_x^{1-\frac{2}{n}, \frac{6n^2}{3n^2+4n-12}})} \leq \epsilon \quad (2.7)$$

for some $0 < \epsilon < \epsilon_1$, where ϵ_1 is some constant $\epsilon_1 = \epsilon_1(E, E', M) > 0$.

We conclude that there exists a solution u to (1.1) on $I \times \mathbb{R}^d$ with the specified initial data $u(t_0)$ at t_0 , and

$$\begin{aligned} \|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I \times \mathbb{R}^n)} &\leq C(M, E, E'), \quad \|u\|_{\dot{S}^1(I)} \leq C(M, E, E'). \\ \|u - \tilde{u}\|_{L_t^6 L_x^{\frac{6n}{3n-8}}} + \|u - \tilde{u}\|_{L_t^3(I; \dot{H}_x^{1-\frac{2}{n}, \frac{6n^2}{3n^2-4n-12}})} &\leq C(M, E, E')\epsilon. \end{aligned}$$

Remark 2.2. As discussions in induction, checking condition (2.7) is more convinient than checking condition (2.3) as one deal with the interaction between the low and high frequency. It plays an essential role to deal with the nonlocal interaction like Hartree equation, etc. Refer to details in Section 4.

We end this section with a few related results. First, if a solution cannot be continued strongly beyond a time T_* , then the $L_t^6 L_x^{\frac{6n}{3n-8}}$ norm must blow up near that time.

Lemma 2.5 (Standard blow-up criterion, [25]). *Let $u_0 \in \dot{H}^1$, and let u be a strong solution to (1.1) on the slab $[t_0, T_0) \times \mathbb{R}^n$ such that*

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}([t_0, T_0) \times \mathbb{R}^n)} < \infty.$$

Then there exists $\delta > 0$ such that the solution u extends to a strong solution to (1.1) on the slab $[t_0, T_0 + \delta] \times \mathbb{R}^n$.

Last, once we have $L_t^6 L_x^{\frac{6n}{3n-8}}$ control of a finite energy solution, we can control all Strichartz norms as well by the standard argument (partition the time interval), .

Lemma 2.6 (Persistence of regularity). *Let $s \geq 0$, I be a compact time interval, and let u be a finite energy solution to (1.1) on $I \times \mathbb{R}^n$ obeying the bounds*

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I \times \mathbb{R}^n)} \leq M.$$

Then, if $t_0 \in I$ and $u(t_0) \in \dot{H}^s$, we have

$$\|u\|_{\dot{H}^s(I \times \mathbb{R}^n)} \leq C(M, E(u)) \|u(t_0)\|_{\dot{H}^s}.$$

3 Overview of proof of global spacetime bounds

We now outline the proof of Theorem 1.1, breaking it down into a number of smaller Propositions, which are the same as in [6], [35], see also [31] and [36]. On one hand, note that the non-local interaction of Hartree equation, we have to use the modified long time perturbation to establish the frequency localization of minimal energy blow up solutions, instead of the classical long time perturbation and bilinear estimate. On the other hand, we obtain the spatial concentration of minimal energy blow up solution after we prove that $L_x^{\frac{2n}{n-2}}$ -norm of minimal energy blow up solutions is bounded from below, which is stronger than the statement that the potential energy of minimal energy blow up solution is bounded from below

3.1 Zeroth stage: Induction on energy

We say that a solution u to (1.1) is *Schwartz* on a slab $I \times \mathbb{R}^n$ if $u(t)$ is a Schwartz function for all $t \in I$. Note that such solutions are then also smooth in time as well as space, thanks to (1.1).

The first observation is that it suffices to do so for Schwartz solutions in order to prove Theorem 1.1. For every energy $E \geq 0$, we define the quantity $0 \leq S(E) \leq +\infty$ by

$$S(E) := \sup \left\{ \|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I_* \times \mathbb{R}^n)} \right\}$$

where the supreme is taken over all compact interval $I_* \subset \mathbb{R}$, and over all Schwartz solution u to (1.1) on $I_* \times \mathbb{R}^n$ with $E(u) \leq E$. We shall adopt the convention that $S(E) = 0$ for $E < 0$.

From the local well-posedness theory, we know that (1.1) is locally wellposedness in \dot{H}^1 . Moreover, from the global well-posedness theory for small initial data, we see that $S(E)$ is finite for small energy E . Our task is to show that

$$S(E) < \infty, \text{ for all } E > 0$$

Assume that $S(E)$ is not always finite. From Lemma 2.3, we see that the set $\{E : S(E) < \infty\}$ is open. Clearly it is also connected and contains 0. By our contradiction hypothesis, there must therefore exist a critical energy $0 < E_{crit} < \infty$ such that $S(E_{crit}) = +\infty$, but $S(E) < \infty$ for all $E < E_{crit}$. One can think of E_{crit} as the minimal energy required to create a blowup solution. From the definition of E_{crit} , the local well-posedness theory, and Lemma 2.6, we have

Lemma 3.1 (Induction on energy hypothesis). *Let $t_0 \in \mathbb{R}$, and let $v(t_0)$ be a Schwartz function such that $E(v(t_0)) \leq E_{crit} - \eta$ for some $\eta > 0$. Then there exists a Schwartz global solution $v : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}$ to (1.1) with initial data $v(t_0)$ at time t_0 such that*

$$\|v\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(\mathbb{R} \times \mathbb{R}^n)} \leq S(E_{crit} - \eta) = C(\eta).$$

Furthermore we have $\|v\|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^n)} \leq C(\eta)$.

For the contradiction argument, we will use six such parameters

$$1 \gg \eta_0 \gg \eta_1 \gg \eta_2 \gg \eta_3 \gg \eta_4 \gg \eta_5 > 0$$

Specifically, we will need a small parameter $0 < \eta_0 = \eta_0(E_{crit}) \ll 1$ depending on E_{crit} . Then we need a smaller quantity $0 < \eta_1 = \eta_1(\eta_0, E_{crit}) \ll 1$ assumed sufficiently small depending on E_{crit} and η_0 . We continue in this fashion, choosing each $0 < \eta_j \ll 1$ to be sufficiently small depending on all previous quantities $\eta_0, \dots, \eta_{j-1}$ and the energy E_{crit} , all the way down to η_5 which is extremely small, much smaller than any quantity depending on $E_{crit}, \eta_0, \dots, \eta_4$ that will appear in our argument. We will always assume implicitly that each η_j has been chosen to be sufficiently small depending on the previous parameters. We will often display the dependence of constants on a parameter, e. g. $C(\eta)$ denotes a large constant depending on η , and $c(\eta)$ will denote a small constant depending upon η . When $\eta_1 \gg \eta_2$, we will understand $c(\eta_1) \gg c(\eta_2)$ and $C(\eta_1) \ll C(\eta_2)$.

Since $S(E_{crit})$ is infinite, it is in particular larger than $\frac{1}{\eta_5}$. By definition of S , this means that we may find a compact interval $I_* \subset \mathbb{R}$ and a smooth solution $u : I_* \times \mathbb{R}^n \rightarrow \mathbb{C}$ to (1.1) with $\frac{E_{crit}}{2} \leq E(u) \leq E_{crit}$ so that u is ridiculously large in the sense that

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I_* \times \mathbb{R}^n)} \geq \frac{1}{\eta_5}. \quad (3.1)$$

We will show that this leads to a contradiction. Although u does not actually blow up, it is still convenient to think of u as almost blowing up in $L_t^6 L_x^{\frac{6n}{3n-8}}$ in the sense of (3.1).

Definition 3.1 (Definition of the minimal energy blowup solution). *A minimal energy blowup solution of (1.1) is a Schwartz solution on a time interval I_* with energy*

$$\frac{1}{2}E_{crit} \leq E(u(t)) \leq E_{crit} \quad (3.2)$$

and $L_t^6 L_x^{\frac{6n}{3n-8}}$ norm enormous in sense of (3.1).

We remark that both conditions (3.1) and (3.2) are invariant under the scaling (1.4). Thus applying the scaling (1.4) to a minimal energy blowup solution produces another minimal energy blowup solution. Some proofs of the sub-proposition below will revolve around a specific frequency N . Henceforth we will not mention the E_{crit} dependence of our constants explicitly, even though all our constants will depend on E_{crit} . We shall need however to keep careful track of the dependence of our argument on η_0, \dots, η_5 . Broadly speaking, we will start with the largest η , namely η_0 , and slowly “retreat” to increasingly smaller values of η as the argument progresses. However we will only retreat as far as η_4 , not η_5 , so that (3.1) will eventually lead to a contradiction when we show that

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I_* \times \mathbb{R}^n)} \leq C(\eta_0, \dots, \eta_4).$$

Together with our assumption that we are considering a minimal energy blowup solution u as in Definition 3.1, the Hardy-Littlewood-Sobolev inequality implies the bounds on kinetic energy

$$\|u\|_{L_t^\infty \dot{H}_x^1(I_* \times \mathbb{R}^n)} \sim 1, \quad (3.3)$$

and potential energy

$$\sup_{t \in I_*} \iint \frac{1}{|x-y|^4} |u(t,x)|^2 |u(t,y)|^2 dx dy \lesssim 1. \quad (3.4)$$

Having displayed our preliminary bounds on the kinetic and potential energy, we briefly discuss the mass

$$\int |u(t,x)|^2 dx,$$

which is another conserved quantity. From (3.3) and the Bernstein inequality, we know that the high frequencies of u have small mass:

$$\|P_{>M} u\|_{L^2} \lesssim \frac{1}{M} \text{ for all } M \in 2^{\mathbb{Z}}. \quad (3.5)$$

Thus we will still be able to use the concept of mass in our estimates as long as we restrict our attention to sufficiently high frequencies.

3.2 First stage: Frequency localization and spatial concentration

We aim to show that a minimal energy blowup solution as Definition 3.1 does not exist. Intuitively, it seems reasonable to expect that a minimal-energy blow up solution should be “irreducible” in the sense that it cannot be decoupled into two or more components of strictly smaller energy that essentially do not interact with each other, since one of the components must then also blow up, contradicting the minimal-energy hypothesis. In particular, we expect at every time that such a solution should be localized in frequency and have the spatial concentration result, which are inspired by those in [6], [31] and [35].

Proposition 3.1 (Frequency delocalization implies spacetime bound). *Let $\eta > 0$, and suppose there exists a dyadic frequency $N_{lo} > 0$ and a time $t_0 \in I_*$ such that we have the energy separation conditions*

$$\|P_{\leq N_{lo}} u(t_0)\|_{\dot{H}^1(\mathbb{R}^n)} \geq \eta \quad (3.6)$$

and

$$\|P_{\geq K(\eta)N_{lo}} u(t_0)\|_{\dot{H}^1(\mathbb{R}^n)} \geq \eta. \quad (3.7)$$

If $K(\eta)$ is sufficiently large depending on η , i.e.

$$K(\eta) \gg C(\eta).$$

Then we have

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I_* \times \mathbb{R}^n)} \leq C(\eta). \quad (3.8)$$

The basic idea is as above discussion, the main tool we need is the modified long time perturbation, which replace the role of the classical long time perturbation and bilinear estimate. See details in Section 4.

Clearly the conclusion of Proposition 3.1 is in conflict with the hypothesis (3.1), and so we should expect the solution to be localized in frequency for every time t . This is indeed the case:

Corollary 3.1 (Frequency localization of energy at each time). *A minimal energy blowup solution of (1.1) satisfies: For every time $t \in I_*$, there exists a dyadic frequency $N(t) \in 2^{\mathbb{Z}}$ such that for every $\eta_4 \leq \eta \leq \eta_0$, we have small energy at frequencies $\ll N(t)$,*

$$\|P_{\leq c(\eta)N(t)} u(t)\|_{\dot{H}^1} \leq \eta, \quad (3.9)$$

small energy at frequencies $\gg N(t)$,

$$\|P_{\geq C(\eta)N(t)}u(t)\|_{\dot{H}^1} \leq \eta. \quad (3.10)$$

and large energy at frequencies $\sim N(t)$,

$$\|P_{c(\eta)N(t) < \cdot < C(\eta)N(t)}u(t)\|_{\dot{H}^1} \sim 1. \quad (3.11)$$

Here $0 < c(\eta) \ll 1 \ll C(\eta) < \infty$ are quantities depending on η .

Proof: See Corollary 4.4 in [6], [31], [35] and [36].

Having shown that a minimal energy blowup solution must be localized in frequency, we turn our attention to space. In physical space, we will not need the full strength of a localization result (Proposition 4.7 in [6]). We will settle instead for a weaker property concerning the spatial concentration of a minimal energy blowup solution. To derive it, we use an idea of [1], [6], [31] and [35], and restrict our analysis to a subinterval $I_0 \subset I_*$. We need to use both the frequency localization result and the fact that the $L_x^{\frac{2n}{n-2}}$ -norm of a minimal energy blowup solution is bounded away from zero in order to prove spatial concentration.

Since u is Schwartz, we may divide the interval I_* into three consecutive pieces $I_* = I_- \cup I_0 \cup I_+$ where each of the three intervals contains a third of the $L_t^6 L_x^{\frac{6n}{3n-8}}$ norm:

$$\int_I \left| \int_{\mathbb{R}^n} |u(t, x)|^{\frac{6n}{3n-8}} dx \right|^{\frac{3n-8}{n}} dt = \frac{1}{3} \int_{I_*} \left| \int_{\mathbb{R}^n} |u(t, x)|^{\frac{6n}{3n-8}} dx \right|^{\frac{3n-8}{n}} dt \quad \text{for } I = I_-, I_0, I_+.$$

In particular from (3.1) we have

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I \times \mathbb{R}^n)} \gtrsim \frac{1}{\eta_5} \quad \text{for } I = I_-, I_0, I_+. \quad (3.12)$$

Thus to contradict (3.1), it suffices to obtain $L_t^6 L_x^{\frac{6n}{3n-8}}$ bounds on one of the three intervals I_-, I_0, I_+ .

It is in the middle interval I_0 that we can obtain physical space concentration; this shall be done in two stages. The first step is to ensure that the norm $\|u(t)\|_{L_x^{\frac{2n}{n-2}}(\mathbb{R}^n)}$ is bounded from below.

Proposition 3.2 ($L_x^{\frac{2n}{n-2}}$ -norm bounded from below). *For any minimal energy blowup solution to (1.1) and all $t \in I_0$, we have*

$$\|u(t)\|_{L_x^{\frac{2n}{n-2}}} \geq \eta_1. \quad (3.13)$$

This is proved in Section 5. Using (3.13) and some simple Fourier analysis as in [6], [31] and [35], we can thus establish the following concentration result:

Proposition 3.3 (Spatial concentration of energy at each time). *For any minimal energy blowup solution to (1.1) and for each $t \in I_0$, there exists $x(t) \in \mathbb{R}^n$ such that*

$$\int_{|x-x(t)| \leq C(\eta_1)/N(t)} |\nabla u(t, x)|^2 dx \gtrsim c(\eta_1) \quad (3.14)$$

and

$$\int_{|x-x(t)| \leq C(\eta_1)/N(t)} |u(t, x)|^p dx \gtrsim c(\eta_1) N(t)^{\frac{n-2}{2}p-n} \quad (3.15)$$

for all $1 < p < \infty$, where the implicit constants depend on p .

Similar result was obtained in [25] in the radial case; To summarize, the statements above tell us that any minimal energy blowup solution to the equation (1.1) must be localized in frequency space at every time and have spatial concentration result at every time. We are still far from done: we have not yet precluded blowup solutions in finite time, nor have we eliminated soliton or soliton-like solutions and double low to high frequency cascade solutions. To achieve this we need spacetime integrability bounds on u . Our main tool for this is a frequency localized interaction Morawetz estimate.

3.3 Second stage: Frequency localized Morawetz estimate

From Bernstein estimate, we have

$$\|P_{c(\eta_0)N(t) < \cdot} P_{c(\eta_0)N(t)} u(t)\|_{\dot{H}^1} \leq C(\eta_0)N(t)\|u\|_{L^\infty L_x^2}. \quad (3.16)$$

Comparing this with (3.11), we obtain the lower bound

$$N(t) \geq c(\eta_0)\|u\|_{L^\infty L_x^2}^{-1} \quad \text{for } t \in I_0.$$

Similar analysis as in [35], we know that the quantity

$$N_{min} := \inf_{t \in I_0} N(t)$$

is strictly positive.

From (3.9) we see that the low frequency portion of the solution has small energy; one can use Strichartz estimates to obtain some spacetime control on those low frequencies. However, we do not yet have much control on the high frequencies, apart from the energy bounds (3.3).

Our initial spacetime bound in the high frequencies is provided by the following interaction Morawetz estimate.

Proposition 3.4 (Frequency-localized interaction Morawetz estimate). *Assuming u is a minimal energy blowup solution of (1.1), and $N_* < c(\eta_2)N_{min}$. Then we have*

$$\iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{\geq N_*}(t, y)|^2 |u_{\geq N_*}(t, x)|^2}{|x - y|^3} dx dy dt \lesssim \eta_1 N_*^{-3}. \quad (3.17)$$

This proposition is proven in Section 6. It is based on the interaction Morawetz inequality developed in [5], [6], [31] and [35]. The key thing about this estimate is that the right-hand side does not depend on I_0 , thus it is useful in eliminating soliton or pseudosolitons, at least for frequencies close to N_{min} .

Moreover, we also obtain Proposition 6.3 during the proof of Proposition 3.4, which is Strichartz control on low and high frequencies of the minimal energy blowup solution. By meaning of scaling and Proposition 6.3, we obtain the following:

Corollary 3.2. *Let $n \geq 5$, u a minimal energy blowup solution to (1.1), and $N_* < c(\eta_2)N_{min}$. Then, we have*

$$\|P_{\geq N_*} u\|_{L^3 L^{\frac{6n}{3n-4}}(I_0 \times \mathbb{R}^n)} \lesssim \eta_1^{\frac{1}{3}} N_*^{-1}. \quad (3.18)$$

Proof: The claim follows interpolating between

$$\nabla P_{\geq N_*} u \in L_t^\infty L_x^2(I_0 \times \mathbb{R}^n)$$

and

$$P_{\geq N_*} u \in L_t^2 L_x^{\frac{2n}{n-2}}(I_0 \times \mathbb{R}^n)$$

which comes from Proposition 6.3.

Combining (3.18) with Proposition 3.3, we obtain the following integral bound on $N(t)$.

Corollary 3.3. *For any minimal energy blowup solutions of (1.1), we have*

$$\int_{I_0} (N(t))^{-1} dt \lesssim C(\eta_1, \eta_2) N_{min}^{-3}.$$

Proof: By (3.18), we have

$$\int_{I_0} \left(\int_{\mathbb{R}^n} |P_{\geq N_*} u|^{\frac{6n}{3n-4}} dx \right)^{\frac{3n-4}{2n}} dt \lesssim \eta_1 N_*^{-3}$$

for all $N_* \leq c(\eta_2) N_{min}$. Let $N_* = c(\eta_2) N_{min}$, then

$$\int_{I_0} \left(\int_{\mathbb{R}^n} |P_{\geq N_*} u|^{\frac{6n}{3n-4}} dx \right)^{\frac{3n-4}{2n}} dt \lesssim C(\eta_1, \eta_2) N_{min}^{-3}. \quad (3.19)$$

On the other hand, by the Bernstein estimate and the conservation of energy, we have

$$\begin{aligned} \int_{|x-x(t)| \leq \frac{C(\eta_1)}{N(t)}} |P_{< N_*} u(t)|^{\frac{6n}{3n-4}} dx &\lesssim C(\eta_1) N(t)^{-n} \|P_{< N_*} u(t)\|_{L_x^\infty}^{\frac{6n}{3n-4}} \\ &\lesssim C(\eta_1) N(t)^{-n} N(t)^{\frac{3n(n-2)}{3n-4}} c(\eta_2) \|P_{< N_*} u(t)\|_{L_x^{\frac{2n}{n-2}}}^{\frac{6n}{3n-4}} \\ &\lesssim c(\eta_2) N(t)^{-\frac{2n}{3n-4}} \|P_{< N_*} u(t)\|_{\dot{H}_x^1}^{\frac{6n}{3n-4}} \lesssim c(\eta_2) N(t)^{-\frac{2n}{3n-4}}. \end{aligned} \quad (3.20)$$

By Proposition 3.3, we also have

$$\int_{|x-x(t)| \leq \frac{C(\eta_1)}{N(t)}} |u(t)|^{\frac{6n}{3n-4}} dx \gtrsim c(\eta_1) N(t)^{-\frac{2n}{3n-4}}. \quad (3.21)$$

Combining (3.20), (3.21) and using the triangle inequality, we have

$$\int_{|x-x(t)| \leq \frac{C(\eta_1)}{N(t)}} |P_{\geq N_*} u(t)|^{\frac{6n}{3n-4}} dx \gtrsim c(\eta_1) N(t)^{-\frac{2n}{3n-4}}.$$

Integrating over I_0 and comparing with (3.19), we get the desired result.

This corollary allows us to obtain some useful bounds in the case when $N(t)$ is bounded from above.

Corollary 3.4 (Nonconcentration implies spacetime bound). *Let $I \subseteq I_0$, and suppose there exists an $N_{max} > 0$ such that $N(t) \leq N_{max}$ for all $t \in I$. Then for any minimal energy blowup solution of (1.1), we have*

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I \times \mathbb{R}^n)} \lesssim C(\eta_0, \eta_1, \eta_2, N_{max}/N_{min}),$$

and furthermore

$$\|u\|_{\dot{S}^1(I \times \mathbb{R}^n)} \lesssim C(\eta_0, \eta_1, \eta_2, N_{max}/N_{min}).$$

Proof: We will prove it by stability theory. First we may use scale invariance (1.4) to rescale $N_{min} = 1$. From Corollary 3.3, we obtain the useful bound

$$|I_0| \lesssim C(\eta_1, \eta_2, N_{max}).$$

Let $\delta = \delta(\eta_0, N_{max}) > 0$ be a small number to be chosen later. Partition I_0 into $O(\frac{|I_0|}{\delta})$ subintervals I_1, \dots, I_J with $|I_j| \leq \delta$. Let $t_j \in I_j$. Since $N(t_j) \leq N_{max}$, Corollary 3.1 yields

$$\|P_{\geq C(\eta_0) N_{max}} u(t_j)\|_{\dot{H}_x^1} \leq \eta_0.$$

Let $\tilde{u}(t) = e^{i(t-t_j)\Delta} P_{<C(\eta_0)N_{max}} u(t_j)$ be the free evolution of the low and medium frequencies of $u(t_j)$. Then we have

$$\|u(t_j) - \tilde{u}(t_j)\|_{\dot{H}_x^1} \leq \eta_0.$$

Moreover, by Remark 2.1, we have

$$\left(\sum_N \|P_N \nabla e^{i(t-t_j)\Delta} (u(t_j) - \tilde{u}(t_j))\|_{L_t^6 L_x^{\frac{6n}{3n-2}}}^2 \right)^{\frac{1}{2}} + \left(\sum_N \|P_N \nabla e^{i(t-t_j)\Delta} (u(t_j) - \tilde{u}(t_j))\|_{L_t^3 L_x^{\frac{6n}{3n-4}}}^2 \right)^{\frac{1}{2}} \lesssim \eta_0.$$

By the Bernstein estimate, Sobolev embedding, and conservation of energy, we obtain

$$\begin{aligned} \|\tilde{u}(t)\|_{L_x^{\frac{6n}{3n-8}}} &\lesssim C(\eta_0, N_{max}) \|\tilde{u}(t)\|_{L_x^{\frac{2n}{n-2}}} \lesssim C(\eta_0, N_{max}) \|\tilde{u}(t)\|_{\dot{H}_x^1} \\ &\lesssim C(\eta_0, N_{max}) \|u(t_j)\|_{\dot{H}_x^1} \lesssim C(\eta_0, N_{max}) \end{aligned}$$

for all $t \in I_j$, so

$$\|\tilde{u}(t)\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I_j \times \mathbb{R}^n)} \lesssim C(\eta_0, N_{max}) \delta^{\frac{1}{6}}.$$

Similarly, we have

$$\begin{aligned} \|\nabla(|\nabla|^{-(n-4)} |\tilde{u}(t)|^2 \tilde{u}(t))\|_{L_x^{\frac{6n}{3n+4}}} &\lesssim \|\nabla \tilde{u}(t)\|_{L_x^{\frac{6n}{3n-4}}} \|\tilde{u}(t)\|_{L_x^{\frac{6n}{3n-8}}}^2 \\ &\lesssim C(\eta_0, N_{max}) \|\nabla \tilde{u}(t)\|_{L_x^2} \|\tilde{u}(t)\|_{L_x^{\frac{6n}{3n-8}}}^2 \\ &\lesssim C(\eta_0, N_{max}) \|\tilde{u}(t)\|_{\dot{H}_x^1}^3 \lesssim C(\eta_0, N_{max}), \end{aligned}$$

which shows that

$$\|\nabla(|\nabla|^{-(n-4)} |\tilde{u}(t)|^2 \tilde{u}(t))\|_{L^{\frac{3}{2}} L_x^{\frac{6n}{3n+4}}(I_j \times \mathbb{R}^n)} \lesssim C(\eta_0, N_{max}) \delta^{\frac{2}{3}},$$

Therefore, Lemma 2.3 with $e = -(|\nabla|^{-(n-4)} |\tilde{u}|^2) \tilde{u}$ implies that

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I_j \times \mathbb{R}^n)} \lesssim 1,$$

provided δ and η_0 are chosen small enough. Summing these bounds in j , we obtain

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I \times \mathbb{R}^n)} \lesssim \frac{|I|}{\delta} \lesssim \frac{|I_0|}{\delta} \lesssim C(\eta_0, \eta_1, \eta_2, N_{max}).$$

The \dot{S}^1 bound then follows from Lemma 2.6.

This above corollary gives the desired contradiction to (3.12) when N_{max}/N_{min} is bounded, i.e., $N(t)$ stays in a bounded range.

3.4 Third stage: Nonconcentration of energy

Now, we will make use of almost conservation law of frequency localized mass to show that any minimal energy blowup solution cannot concentrate energy to very high frequencies. Instead the solution always leaves a nontrivial amount of mass and energy behind at medium frequencies. This “littering” of the solution will serve to keep $N(t)$ from escaping to infinity, which is inspired by the ideas in [6], [31] and [35]. We will prove in Section 7.

Proposition 3.5 (Energy cannot evacuate from low frequencies). *For any minimal energy blowup solution of (1.1), we have*

$$N(t) \leq C(\eta_4)N_{min}$$

for all $t \in I_0$

By combining Proposition 3.5 with Corollary 3.4, we encounter a contradiction to (3.12) which completes the proof of Theorem 1.1.

4 Frequency delocalization at one time implies spacetime bound

Using the modified long time perturbation, we now prove Proposition 3.1 as in [6], [31] and [35]. Let $0 < \epsilon = \epsilon(\eta) \ll 1$ be a small number to be chosen later. If $K(\eta)$ is sufficiently large depending on ϵ , then one can find ϵ^{-2} disjoint intervals $[\epsilon^2 N_j, \epsilon^{-2} N_j]$ contained in $[N_{lo}, K(\eta)N_{lo}]$. By (3.3) and the pigeonhole principle, we may find an N_j such that the localization of $u(t_0)$ to the interval $[\epsilon^2 N_j, \epsilon^{-2} N_j]$ has very little energy:

$$\|P_{\epsilon^2 N_j \leq \cdot \leq \epsilon^{-2} N_j} u(t_0)\|_{\dot{H}_x^1} \lesssim \epsilon. \quad (4.1)$$

Since both the statement and conclusion of the proposition are invariant under the scaling (1.4), we normalize $N_j = 1$.

Define

$$u_{lo}(t_0) = P_{\leq \epsilon} u(t_0), \quad u_{hi}(t_0) = P_{\geq 1/\epsilon} u(t_0).$$

We claim that u_{hi} and u_{lo} have smaller energy than u .

Lemma 4.1. *If ϵ is sufficiently small depending on η , we have*

$$E(u_{lo}(t_0)), E(u_{hi}(t_0)) \leq E_{crit} - c\eta^C.$$

Proof: Without loss of generality, we will prove this for u_{lo} , the proof for u_{hi} is similar. Define

$$u_{hi'}(t_0) = P_{> \epsilon} u(t_0),$$

so that $u(t_0) = u_{lo}(t_0) + u_{hi'}(t_0)$ and consider the quantity

$$|E(u(t_0)) - E(u_{lo}(t_0)) - E(u_{hi'}(t_0))|.$$

By the definition of energy, we can bound this by

$$\begin{aligned} & \left| \langle \nabla u_{lo}(t_0), \nabla u_{hi'}(t_0) \rangle \right| + \left| \int \left(|\nabla|^{-(n-4)} |u(t_0)|^2 |u(t_0)|^2 \right. \right. \\ & \quad \left. \left. - |\nabla|^{-(n-4)} |u_{lo}(t_0)|^2 |u_{lo}(t_0)|^2 - |\nabla|^{-(n-4)} |u_{hi'}(t_0)|^2 |u_{hi'}(t_0)|^2 \right) dx \right|. \end{aligned} \quad (4.2)$$

We first deal with the kinetic energy. By the Bernstein estimate, (3.3) and (4.1), We have

$$\|u_{hi'}(t_0)\|_{L_x^2} \lesssim \sum_{N > \epsilon} \|P_N u(t_0)\|_{L_x^2} \lesssim \sum_{\epsilon < N \leq \epsilon^{-2}} N^{-1} \epsilon + \sum_{N > \epsilon^{-2}} N^{-1} \lesssim 1, \quad (4.3)$$

therefore,

$$\begin{aligned} \left| \langle \nabla u_{lo}(t_0), \nabla u_{hi'}(t_0) \rangle \right| & \lesssim \left| \langle \nabla P_{> \epsilon} P_{\leq \epsilon} u(t_0), \nabla u(t_0) \rangle \right| \lesssim \|\nabla P_{> \epsilon} P_{\leq \epsilon} u(t_0)\|_{L_x^2} \|\nabla u(t_0)\|_{L_x^2} \\ & \lesssim \|\xi \varphi(\xi/\epsilon) (1 - \varphi(\xi/\epsilon)) \widehat{u(t_0)}(\xi)\|_{L_x^2} \lesssim \epsilon \|u_{hi'}(t_0)\|_{L_x^2} \lesssim \epsilon. \end{aligned} \quad (4.4)$$

Next we deal with the potential energy part of (4.2). By the Bernstein inequality, (3.3) and (4.1), We have

$$\begin{aligned}
\|u_{hi'}(t_0)\|_{L_x^{\frac{3n}{2n-4}}} &\lesssim \sum_{N \geq \epsilon} \|P_N u\|_{L_x^{\frac{3n}{2n-4}}} \lesssim \sum_{N \geq \epsilon} N^{-1} N^{\frac{n}{2} - \frac{2n-4}{3}} \|\nabla P_N u\|_{L_x^2} \\
&\lesssim \sum_{\epsilon \leq N \leq \epsilon^{-2}} N^{-1} N^{\frac{n}{2} - \frac{2n-4}{3}} \epsilon + \sum_{N \geq \epsilon^{-2}} N^{-1} N^{\frac{n}{2} - \frac{2n-4}{3}} \lesssim \epsilon^{\frac{8-n}{6}}, \\
\|u_{hi'}(t_0)\|_{L_x^{\frac{n}{n-2}}} &\lesssim \sum_{N \geq \epsilon} \|P_N u\|_{L_x^{\frac{n}{n-2}}} \lesssim \sum_{N \geq \epsilon} N^{-1} N^{\frac{n}{2} - n + 2} \|\nabla P_N u\|_{L_x^2} \\
&\lesssim \sum_{\epsilon \leq N \leq \epsilon^{-2}} N^{-1} N^{\frac{n}{2} - n + 2} \epsilon + \sum_{N \geq \epsilon^{-2}} N^{-1} N^{\frac{n}{2} - n + 2} \lesssim \epsilon^{2 - \frac{n}{2}}, \\
\|u_{lo}(t_0)\|_{L_x^\infty} &\lesssim \epsilon^{\frac{n-2}{2}} \|u_{lo}(t_0)\|_{L_x^{\frac{2n}{n-2}}} \lesssim \epsilon^{\frac{n-2}{2}} \|u_{lo}(t_0)\|_{\dot{H}_x^1} \lesssim \epsilon^{\frac{n-2}{2}} \\
\|u_{lo}(t_0)\|_{L_x^{\frac{6n}{3n-8}}} &\lesssim \epsilon^{\frac{n-2}{2} - \frac{3n-8}{6}} \|u_{lo}(t_0)\|_{L_x^{\frac{2n}{n-2}}} \lesssim \epsilon^{\frac{1}{3}},
\end{aligned}$$

combining the above estimates with (4.3), we obtain

$$\begin{aligned}
&\int \left| |\nabla|^{-(n-4)} |u(t_0)|^2 |u(t_0)|^2 - |\nabla|^{-(n-4)} |u_{lo}(t_0)|^2 |u_{lo}(t_0)|^2 - |\nabla|^{-(n-4)} |u_{hi'}(t_0)|^2 |u_{hi'}(t_0)|^2 \right| dx \\
&\lesssim \int |\nabla|^{-(n-4)} |u_{lo}(t_0)|^2 (|u_{lo}(t_0)| + |u_{hi'}(t_0)|) |u_{hi'}(t_0)| dx \\
&\quad + \int |\nabla|^{-(n-4)} |u_{lo}(t_0) u_{hi'}(t_0)| (|u_{lo}(t_0)|^2 + |u_{hi'}(t_0)|^2) dx \\
&\quad + \int |\nabla|^{-(n-4)} |u_{hi'}(t_0)|^2 (|u_{lo}(t_0)| + |u_{hi'}(t_0)|) |u_{lo}(t_0)| dx \\
&\lesssim \|u_{hi'}\|_{L_x^{\frac{3n}{2n-4}}}^3 \|u_{lo}\|_{L_x^\infty} + \|u_{hi'}\|_{L_x^{\frac{n}{n-2}}}^2 \|u_{lo}\|_{L_x^\infty}^2 + \|u_{hi'}\|_{L_x^2} \|u_{lo}\|_{L_x^{\frac{6n}{3n-8}}}^3 \\
&\lesssim \left(\epsilon^{\frac{8-n}{6}} \right)^3 \epsilon^{\frac{n-2}{2}} + \left(\epsilon^{2 - \frac{n}{2}} \right)^2 \left(\epsilon^{\frac{n-2}{2}} \right)^2 + \left(\epsilon^{\frac{1}{3}} \right)^3 \lesssim \epsilon.
\end{aligned} \tag{4.5}$$

Therefore, we have from (4.4) and (4.5)

$$|E(u(t_0)) - E(u_{lo}(t_0)) - E(u_{hi'}(t_0))| \lesssim \epsilon.$$

Since

$$E(u) \leq E_{crit},$$

and by hypothesis (3.7), we have

$$E(u_{hi'}(t_0)) \gtrsim \|u_{hi'}(t_0)\|_{\dot{H}_x^1}^2 \gtrsim \eta^2,$$

the triangle inequality implies

$$E(u_{lo}(t_0)) \leq E_{crit} - c\eta^C,$$

provided we choose ϵ sufficiently small. Similarly, one proves

$$E(u_{hi}(t_0)) \leq E_{crit} - c\eta^C.$$

Now, since

$$E(u_{lo}(t_0)) \leq E_{crit} - c\eta^C, \quad E(u_{hi}(t_0)) \leq E_{crit} - c\eta^C.$$

We can apply Lemma 3.1, we know that there exist Schwartz solutions $u_{lo}(t), u_{hi}(t)$ to (1.1) on the slab $I_* \times \mathbb{R}^n$ with initial data $u_{lo}(t_0), u_{hi}(t_0)$ at time t_0 , and furthermore

$$\|u_{lo}\|_{\dot{S}^1(I_* \times \mathbb{R}^n)} \leq C(\eta), \quad \|u_{hi}\|_{\dot{S}^1(I_* \times \mathbb{R}^n)} \leq C(\eta).$$

From Lemma 2.6, we also have

$$\begin{aligned}\|u_{lo}\|_{\dot{S}^{1+s}(I_* \times \mathbb{R}^n)} &\leq C(\eta) \|u_{lo}(t_0)\|_{\dot{H}_x^{1+s}} \leq C(\eta) \epsilon^s, \quad \forall 0 \leq s \leq 1, \\ \|u_{hi}\|_{\dot{S}^{1-k}(I_* \times \mathbb{R}^n)} &\leq C(\eta) \|u_{hi}(t_0)\|_{\dot{H}_x^{1-k}} \leq C(\eta) \epsilon^k, \quad \forall 0 \leq k \leq 1.\end{aligned}$$

Define

$$\tilde{u}(t) := u_{lo}(t) + u_{hi}(t).$$

We claim that $\tilde{u}(t)$ is a near-solution to (1.1).

Lemma 4.2. *We have*

$$i\tilde{u}_t + \Delta \tilde{u} = (|\nabla|^{-(n-4)} |\tilde{u}|^2) \tilde{u} - e$$

where the error e obeys the bound

$$\|e\|_{L_t^{\frac{3}{2}}(I; \dot{H}_x^{1-\frac{2}{n}, \frac{6n^2}{3n^2+4n-12}})} \leq C(\eta) \epsilon^{\frac{4}{n}}. \quad (4.6)$$

Proof: In order to estimate the error term

$$\begin{aligned}e &= |\nabla|^{-(n-4)} |\tilde{u}|^2 \tilde{u} - |\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo} - |\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi} \\ &= |\nabla|^{-(n-4)} |u_{lo}|^2 u_{hi} + 2|\nabla|^{-(n-4)} \operatorname{Re}(u_{lo} \bar{u}_{hi})(u_{lo} + u_{hi}) + |\nabla|^{-(n-4)} |u_{hi}|^2 u_{lo},\end{aligned}$$

we obtain by the Leibniz rule

$$\begin{aligned}\|e\|_{L_t^{\frac{3}{2}}(I; \dot{H}_x^{1-\frac{2}{n}, \frac{6n^2}{3n^2+4n-12}})} &\lesssim \left\| |\nabla|^{-(n-4)} (|\nabla|^{1-\frac{2}{n}} u_{lo} \bar{u}_{lo}) u_{hi} \right\|_{L_t^{\frac{3}{2}}(I; L_x^{\frac{6n^2}{3n^2+4n-12}})} + \left\| |\nabla|^{-(n-4)} |u_{lo}|^2 |\nabla|^{1-\frac{2}{n}} u_{hi} \right\|_{L_t^{\frac{3}{2}}(I; L_x^{\frac{6n^2}{3n^2+4n-12}})} \\ &\quad + \left\| |\nabla|^{-(n-4)} (|\nabla|^{1-\frac{2}{n}} u_{lo} \bar{u}_{hi}) u_{lo} \right\|_{L_t^{\frac{3}{2}}(I; L_x^{\frac{6n^2}{3n^2+4n-12}})} + \left\| |\nabla|^{-(n-4)} (u_{lo} |\nabla|^{1-\frac{2}{n}} \bar{u}_{hi}) u_{lo} \right\|_{L_t^{\frac{3}{2}}(I; L_x^{\frac{6n^2}{3n^2+4n-12}})} \\ &\quad + \left\| |\nabla|^{-(n-4)} (u_{lo} \bar{u}_{hi}) |\nabla|^{1-\frac{2}{n}} u_{lo} \right\|_{L_t^{\frac{3}{2}}(I; L_x^{\frac{6n^2}{3n^2+4n-12}})} + \left\| |\nabla|^{-(n-4)} (|\nabla|^{1-\frac{2}{n}} u_{lo} \bar{u}_{hi}) u_{hi} \right\|_{L_t^{\frac{3}{2}}(I; L_x^{\frac{6n^2}{3n^2+4n-12}})} \\ &\quad + \left\| |\nabla|^{-(n-4)} (u_{lo} |\nabla|^{1-\frac{2}{n}} \bar{u}_{hi}) u_{hi} \right\|_{L_t^{\frac{3}{2}}(I; L_x^{\frac{6n^2}{3n^2+4n-12}})} + \left\| |\nabla|^{-(n-4)} (u_{lo} \bar{u}_{hi}) |\nabla|^{1-\frac{2}{n}} u_{hi} \right\|_{L_t^{\frac{3}{2}}(I; L_x^{\frac{6n^2}{3n^2+4n-12}})} \\ &\quad + \left\| |\nabla|^{-(n-4)} |u_{hi}|^2 |\nabla|^{1-\frac{2}{n}} u_{lo} \right\|_{L_t^{\frac{3}{2}}(I; L_x^{\frac{6n^2}{3n^2+4n-12}})} + \left\| |\nabla|^{-(n-4)} (|\nabla|^{1-\frac{2}{n}} u_{hi} \bar{u}_{hi}) u_{lo} \right\|_{L_t^{\frac{3}{2}}(I; L_x^{\frac{6n^2}{3n^2+4n-12}})} \\ &=: \sum_{i=1}^{10} I_i.\end{aligned}$$

We first deal with the terms which contains $|\nabla|^{1-\frac{2}{n}} u_{lo}$. By the Hardy-Littlewood-Sobolev inequality and Sobolev inequality, we have

$$\begin{aligned}I_1 + I_3 + I_5 &\lesssim \left\| \nabla u_{lo} \right\|_{L_t^6 L_x^{\frac{6n}{3n-8}}} \left\| u_{lo} \right\|_{L_t^6 L_x^{\frac{6n}{3n-8}}} \left\| u_{hi} \right\|_{L_t^3 L_x^{\frac{6n}{3n-4}}} \\ &\lesssim \left\| u_{lo} \right\|_{\dot{S}^2} \left\| u_{lo} \right\|_{\dot{S}^1} \left\| u_{hi} \right\|_{\dot{S}^0} \lesssim C(\eta) \epsilon^2, \\ I_6 + I_9 &\lesssim \left\| u_{lo} \right\|_{\dot{S}^2} \left\| u_{hi} \right\|_{\dot{S}^1} \left\| u_{hi} \right\|_{\dot{S}^0} \\ &\lesssim C(\eta) \epsilon^2,\end{aligned}$$

Next, we deal with the terms which contains $|\nabla|^{1-\frac{2}{n}} u_{hi}$. Using the Hardy-Littlewood-Sobolev inequality, we get

$$\begin{aligned}I_2 + I_4 &\lesssim \left\| |\nabla|^{1-\frac{2}{n}} u_{hi} \right\|_{L_t^6 L_x^{\frac{6n}{3n-8}}} \left\| u_{lo} \right\|_{L_t^3 L_x^{\frac{6n^2}{3n^2-10n-6}}}^2 \\ &\lesssim \left\| u_{hi} \right\|_{\dot{S}^{1-\frac{2}{n}}} \left\| u_{lo} \right\|_{\dot{S}^{1+\frac{1}{n}}}^2 \lesssim C(\eta) \epsilon^{\frac{4}{n}}.\end{aligned} \quad (4.7)$$

It is the place where we must use Lemma 2.4 instead of Lemma 2.3, otherwise we can only obtain the boundedness of I_2, I_4 , no any decay !

The last three estimates follow from the Hardy-Littlewood-Sobolev inequality and Sobolev inequality

$$\begin{aligned} I_7 + I_8 + I_{10} &\lesssim \|\nabla u_{hi}\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \|u_{hi}\|_{L_t^6 L_x^{\frac{6n}{3n-2}}} \|u_{lo}\|_{L_t^\infty L_x^{\frac{2n}{n-4}}} \\ &\lesssim \|u_{hi}\|_{\dot{S}^1} \|u_{hi}\|_{\dot{S}^0} \|u_{lo}\|_{\dot{S}^2} \lesssim C(\eta) \epsilon^2. \end{aligned}$$

Next, we derive estimates on u from those on \tilde{u} via perturbation theory. More precisely, we know from (4.1) that

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}_x^1} \lesssim \epsilon, \quad (4.8)$$

and hence, we have

$$\|e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L_t^3(I; \dot{H}_x^{1-\frac{2}{n}, \frac{6n^2}{3n^2-4n-12}})} \lesssim \epsilon$$

By the Strichartz estimate, we also have that

$$\|\tilde{u}\|_{L^\infty \dot{H}^1(I_* \times \mathbb{R}^n)} \lesssim \|\tilde{u}\|_{\dot{S}^1(I_* \times \mathbb{R}^n)} \lesssim \|u_{lo}\|_{\dot{S}^1(I_* \times \mathbb{R}^n)} + \|u_{hi}\|_{\dot{S}^1(I_* \times \mathbb{R}^n)} \lesssim C(\eta),$$

and hence,

$$\|\tilde{u}\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I_* \times \mathbb{R}^n)} \lesssim \|\tilde{u}\|_{\dot{S}^1(I_* \times \mathbb{R}^n)} \lesssim C(\eta).$$

So in view of (4.6), if ϵ is sufficiently small depending on η , we can apply Lemma 2.4 and obtain the desired bound (3.8). This completes the proof of Proposition 3.1.

5 Small $L_x^{\frac{2n}{n-2}}$ norm implies spacetime bound

We now prove Proposition 3.2. Here $L_x^{\frac{2n}{n-2}}$ norm is not the potential energy of the solution. We will argue by contradiction just as in [6], see also [31] and [35]. Suppose there exists some time $t_0 \in I_0$ such that

$$\|u(t_0)\|_{L_x^{\frac{2n}{n-2}}} < \eta_1. \quad (5.1)$$

Using (1.4), we scale $N(t_0) = 1$. If the linear evolution $e^{i(t-t_0)\Delta}u(t_0)$ had small $L_t^6 L_x^{\frac{6n}{3n-8}}$ -norm, then by perturbation theory, the nonlinear solution would have small $L_t^6 L_x^{\frac{6n}{3n-8}}$ -norm as well. Hence, we may assume

$$\|e^{i(t-t_0)\Delta}u(t_0)\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(\mathbb{R} \times \mathbb{R}^n)} \gtrsim 1.$$

On the other hand, Corollary 3.1 implies that

$$\|P_{lo}u(t_0)\|_{\dot{H}_x^1} + \|P_{hi}u(t_0)\|_{\dot{H}_x^1} \lesssim \eta_0,$$

where $P_{lo} = P_{<c(\eta_0)}$ and $P_{hi} = P_{>C(\eta_0)}$. The Strichartz estimates yield

$$\|e^{i(t-t_0)\Delta}P_{lo}u(t_0)\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(\mathbb{R} \times \mathbb{R}^n)} + \|e^{i(t-t_0)\Delta}P_{hi}u(t_0)\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \eta_0.$$

Thus,

$$\|e^{i(t-t_0)\Delta}P_{med}u(t_0)\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(\mathbb{R} \times \mathbb{R}^n)} \approx 1.$$

where $P_{med} = 1 - P_{lo} - P_{hi}$. However, $P_{med}u(t_0)$ has bounded energy and Fourier support in $c(\eta_0) \lesssim |\xi| \lesssim C(\eta_0)$, an application of the Strichartz and Bernstein estimate yields

$$\|e^{i(t-t_0)\Delta}P_{med}u(t_0)\|_{L_t^{\frac{6(n-2)}{n}}L_x^{\frac{6(n-2)}{3n-8}}(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|P_{med}u(t_0)\|_{L_x^2} \lesssim C(\eta_0).$$

Combining these estimates with the Hölder inequality, we get

$$\|e^{i(t-t_0)\Delta}P_{med}u(t_0)\|_{L_{t,x}^\infty(\mathbb{R}\times\mathbb{R}^n)} \gtrsim c(\eta_0).$$

In particular, there exist a time $t_1 \in \mathbb{R}$ and a point $x_1 \in \mathbb{R}^n$ so that

$$|e^{i(t_1-t_0)\Delta}(P_{med}u(t_0))(x_1)| \gtrsim c(\eta_0). \quad (5.2)$$

We may perturb t_1 such that $t_1 \neq t_0$ and, by time reversal symmetry, we may take $t_1 < t_0$. Let δ_{x_1} be the Dirac mass at x_1 . Define $f(t_1) := P_{med}\delta_{x_1}$ and for $t > t_1$ define $f(t) := e^{i(t-t_1)\Delta}f(t_1)$. We first recall a property about $f(t)$ as in [6].

Lemma 5.1. *For any $t \in \mathbb{R}$ and any $1 \leq p \leq \infty$, we have*

$$\|f(t)\|_{L_x^p} \leq C(\eta_0)\langle t - t_1 \rangle^{\frac{n}{p} - \frac{n}{2}}.$$

From (5.1) and the Hölder inequality, we have

$$|\langle f(t_0), u(t_0) \rangle| \lesssim \|f(t_0)\|_{L_x^{\frac{2n}{n-2}}} \|u(t_0)\|_{L_x^{\frac{2n}{n-2}}} \lesssim \eta_1 C(\eta_0) \langle t_0 - t_1 \rangle.$$

On the other hand, by (5.2), we get

$$|\langle f(t_0), u(t_0) \rangle| = |e^{i(t_1-t_0)\Delta}(P_{med}u(t_0))(x_1)| \gtrsim c(\eta_0).$$

So $\langle t_0 - t_1 \rangle \gtrsim \frac{c(\eta_0)}{\eta_1}$, i.e., t_1 is far from t_0 . In particular, the time t_1 of concentration must be far from t_0 where the $L_x^{\frac{2n}{n-2}}$ -norm is small.

Also, from Lemma 5.1, we have

$$\|f\|_{L_t^6 L_x^{\frac{6n}{3n-2}}([t_0, \infty) \times \mathbb{R}^n)} \lesssim C(\eta_0) \|\langle \cdot - t_1 \rangle^{-\frac{1}{3}}\|_{L^6([t_0, \infty))} \lesssim C(\eta_0) \eta_1^{\frac{1}{6}}, \quad (5.3)$$

$$\|f\|_{L_t^3 L_x^{\frac{6n}{3n-4}}([t_0, \infty) \times \mathbb{R}^n)} \lesssim C(\eta_0) \|\langle \cdot - t_1 \rangle^{-\frac{2}{3}}\|_{L^3([t_0, \infty))} \lesssim C(\eta_0) \eta_1^{\frac{1}{3}}. \quad (5.4)$$

Now we use the induction hypothesis. Split $u(t_0) = v(t_0) + w(t_0)$ where $w(t_0) = \delta e^{i\theta} \Delta^{-1} f(t_0)$ for some small $\delta = \delta(\eta_0) > 0$ and phase θ to be chosen later. One should think of $w(t_0)$ as the contribution coming from the point (t_1, x_1) where the solution concentrates. We will show that for an appropriate choice of δ and θ , $v(t_0)$ has slightly smaller energy than u . By the definition of f and an integration by parts, we have

$$\frac{1}{2} \int_{\mathbb{R}^n} |\nabla v(t_0)|^2 dx \leq E_{crit} + \delta \operatorname{Re} e^{-i\theta} \langle u(t_0), f(t_0) \rangle + O(\delta^2 C(\eta_0)).$$

Choosing δ and θ appropriately, we get

$$\frac{1}{2} \int_{\mathbb{R}^n} |\nabla v(t_0)|^2 dx \leq E_{crit} - c(\eta_0).$$

Also, by Lemma 5.1, we have

$$\|w(t_0)\|_{L_x^{\frac{2n}{n-2}}} \lesssim C(\eta_0) \eta_1.$$

So, by (5.1), the Hardy-Littlewood-Sobolev inequality, and the triangle inequality we obtain

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v(t_0, x)|^2 |v(t_0, y)|^2}{|x - y|^4} dx dy \lesssim \|v(t_0)\|_{L_x^{\frac{2n}{n-2}}}^4 \lesssim \|u(t_0)\|_{L_x^{\frac{2n}{n-2}}}^4 + \|w(t_0)\|_{L_x^{\frac{2n}{n-2}}}^4 \lesssim C(\eta_0) \eta_1^4.$$

Combining the above two energy estimates and taking η_1 sufficiently small depending on η_0 , we obtain

$$E(v(t_0)) \leq E_{crit} - c(\eta_0).$$

Lemma 3.1 implies that there exists a global solution v to (1.1) with initial data $v(t_0)$ at time t_0 satisfying

$$\|v\|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^n)} \lesssim C(\eta_0).$$

In particular,

$$\|v\|_{L_t^\infty \dot{H}_x^1([t_0, \infty) \times \mathbb{R}^n)} + \|v\|_{L_t^6 L_x^{\frac{6n}{3n-8}}([t_0, \infty) \times \mathbb{R}^n)} \lesssim C(\eta_0).$$

Moreover, by the Bernstein estimate,

$$\|w(t_0)\|_{\dot{H}_x^1} \lesssim C(\eta_0).$$

By (5.3), (5.4) and the frequency localization, we estimate

$$\begin{aligned} \sum_N \|P_N \nabla e^{i(t-t_0)\Delta} w(t_0)\|_{L_t^6 L_x^{\frac{6n}{3n-2}}([t_0, \infty) \times \mathbb{R}^n)}^2 &\lesssim \sum_{N \leq C(\eta_0)} \|P_N \nabla e^{i(t-t_0)\Delta} w(t_0)\|_{L_t^6 L_x^{\frac{6n}{3n-2}}([t_0, \infty) \times \mathbb{R}^n)}^2 \\ &\quad + \sum_{N > C(\eta_0)} \|P_N \nabla e^{i(t-t_0)\Delta} w(t_0)\|_{L_t^6 L_x^{\frac{6n}{3n-2}}([t_0, \infty) \times \mathbb{R}^n)}^2 \\ &\lesssim \delta(\eta_0) \sum_{N \leq C(\eta_0)} N^2 C(\eta_0) \|f\|_{L_t^6 L_x^{\frac{6n}{3n-2}}([t_0, \infty) \times \mathbb{R}^n)}^2 \\ &\quad + \delta(\eta_0) \sum_{N > C(\eta_0)} N^{-2} \|f\|_{L_t^6 L_x^{\frac{6n}{3n-2}}([t_0, \infty) \times \mathbb{R}^n)}^2 \\ &\lesssim C(\eta_0) \eta_1^{\frac{1}{3}}, \\ \sum_N \|P_N \nabla e^{i(t-t_0)\Delta} w(t_0)\|_{L_t^3 L_x^{\frac{6n}{3n-4}}([t_0, \infty) \times \mathbb{R}^n)}^2 &\lesssim C(\eta_0) \eta_1^{\frac{2}{3}}. \end{aligned}$$

So, if η_1 is sufficiently small depending on η_0 , we can apply Lemma 2.3 with $\tilde{u} = v$ and $e = 0$ to conclude that u extends to all of $[t_0, \infty)$ and obeys

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}([t_0, \infty) \times \mathbb{R}^n)} \lesssim C(\eta_0, \eta_1).$$

Since $[t_0, \infty)$ contains I_+ , the above estimate contradicts (3.12) if η_5 is chosen sufficiently small. This concludes the proof of Proposition 3.2.

6 Interaction Morawetz inequality

The goal of this section is to prove Proposition 3.4, which is used to eliminate the soliton-like solutions.

6.1 Interaction Morawetz: Generalities

We start by recalling the standard Morawetz action centered at a point. Let a be a function on the slab $I \times \mathbb{R}^n$ and ϕ satisfying

$$i\partial_t \phi + \Delta \phi = \mathcal{N} \quad (6.1)$$

on $I \times \mathbb{R}^n$. We define the Morawetz action centered at zero to be

$$M_a^0(t) = 2 \int_{\mathbb{R}^n} a_j(x) \operatorname{Im}(\bar{\phi}(x) \phi_j(x)) dx$$

where repeated indices are implicitly summed. A simple calculation yields

Lemma 6.1.

$$\partial_t M_a^0 = \int_{\mathbb{R}^n} (-\Delta \Delta a) |\phi|^2 + 4 \int_{\mathbb{R}^n} a_{jk} \operatorname{Re}(\bar{\phi}_j \phi_k) + 2 \int_{\mathbb{R}^n} a_j \{\mathcal{N}, \phi\}_p^j,$$

where we define the momentum bracket to be $\{f, g\}_p = \operatorname{Re}(f \nabla \bar{g} - g \nabla \bar{f})$.

Now let $a(x) = |x|$, easy computations show that in dimension $n \geq 5$ we have the following identities:

$$\begin{aligned} a_j(x) &= \frac{x_j}{|x|}, & a_{jk}(x) &= \frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3}, \\ \Delta a(x) &= \frac{n-1}{|x|}, & -\Delta \Delta a(x) &= \frac{(n-1)(n-3)}{|x|^3}, \end{aligned}$$

and hence,

$$\begin{aligned} \partial_t M_a^0 &= (n-1)(n-3) \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^3} dx + 4 \int_{\mathbb{R}^n} \left(\frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3} \right) \operatorname{Re}(\bar{\phi}_j \phi_k)(x) dx \\ &\quad + 2 \int_{\mathbb{R}^n} \frac{x_j}{|x|} \{\mathcal{N}, \phi\}_p^j(x) dx \\ &= (n-1)(n-3) \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^3} dx + 4 \int_{\mathbb{R}^n} \frac{1}{|x|} |\nabla_0 \phi(x)|^2 dx \\ &\quad + 2 \int_{\mathbb{R}^n} \frac{x_j}{|x|} \{\mathcal{N}, \phi\}_p^j(x) dx \end{aligned}$$

where we use ∇_0 to denote the complement of the radial portion of the gradient.

We may center the above argument at any other point $y \in \mathbb{R}^n$. Choosing $a(x) = |x - y|$, we define the Morawetz action centered at y to be

$$M_a^y(t) = 2 \int_{\mathbb{R}^n} \frac{x - y}{|x - y|} \operatorname{Im}(\bar{\phi}(x) \nabla \phi(x)) dx.$$

The same calculations now yield that

$$\partial_t M_a^y = (n-1)(n-3) \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x - y|^3} dx + 4 \int_{\mathbb{R}^n} \frac{1}{|x - y|} |\nabla_y \phi(x)|^2 dx + 2 \int_{\mathbb{R}^n} \frac{x - y}{|x - y|} \{\mathcal{N}, \phi\}_p(x) dx.$$

We are now ready to define the interaction Morawetz potential:

$$M^{interact}(t) = \int_{\mathbb{R}^n} |\phi(t, y)|^2 M_a^y(t) dy = 2 \operatorname{Im} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\phi(t, y)|^2 \frac{x - y}{|x - y|} \bar{\phi}(t, x) \nabla \phi(t, x) dx dy.$$

One gets immediately the estimate

$$|M^{interact}(t)| \leq 2\|\phi(t)\|_{L_x^2}^3 \|\phi(t)\|_{\dot{H}_x^1}.$$

Calculating the time derivative of the interaction Morawetz potential, we get the following virial-type identity:

$$\begin{aligned} \partial_t M^{interact} &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left((n-1)(n-3) \frac{|\phi(y)|^2 |\phi(x)|^2}{|x-y|^3} + 2|\phi(y)|^2 \frac{x-y}{|x-y|} \{\mathcal{N}, \phi\}_p(x) \right) dx dy \\ &\quad + 4 \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\phi(y)|^2 |\nabla_y \phi(x)|^2}{|x-y|} dx dy + 2 \int_{\mathbb{R}^n} \partial_{y^k} \text{Im}(\phi \bar{\phi}_k)(y) M_a^y dy \\ &\quad + 4 \text{Im} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \{\mathcal{N}, \phi\}_m(y) \frac{x-y}{|x-y|} \nabla \phi(x) \bar{\phi}(x) dx dy. \end{aligned} \quad (6.2)$$

where the mass bracket is defined to be $\{f, g\}_m = \text{Im}(f \bar{g})$.

Similar proof as Proposition 2.5 in [5], see also Proposition 10.3 in [6], Lemma 5.3 in [31] or Proposition 5.5 in [35], we have

Lemma 6.2. $(6.2) \geq 0$.

Thus, integrating the virial-type identity over the compact interval I_0 , we get

Proposition 6.1 (Interaction Morawetz inequality). *Let ϕ be a (Schwartz) solution to the equation*

$$i\partial_t \phi + \Delta \phi = \mathcal{N}$$

on a spacetime slab $I_0 \times \mathbb{R}^n$. Then we have

$$\begin{aligned} &\iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} (n-1)(n-3) \frac{|\phi(t, y)|^2 |\phi(t, x)|^2}{|x-y|^3} + 2|\phi(t, y)|^2 \frac{x-y}{|x-y|} \{\mathcal{N}, \phi\}_p(t, x) dx dy dt \\ &\leq 4 \|\phi\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^n)}^3 \|\phi\|_{L_t^\infty \dot{H}_x^1(I_0 \times \mathbb{R}^n)} + 4 \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |\{\mathcal{N}, \phi\}_m(t, y)| |\nabla \phi(t, x)| |\phi(t, x)| dx dy dt. \end{aligned}$$

6.2 Interaction Morawetz: The setup

We are ready to start the proof of Proposition 3.4. By scaling invariance, we normalize $N_* = 1$ and define

$$u_{lo}(t) = P_{\leq 1} u(t), \quad u_{hi}(t) = P_{> 1} u(t).$$

Since we assume $1 = N_* < c(\eta_2) N_{min}$, we have $1 < c(\eta_2) N(t), \forall t \in I_0$. From Corollary 3.1 and the Sobolev embedding, we have the low frequency estimate

$$\|u_{< \frac{1}{\eta_2}}\|_{L_t^\infty \dot{H}_x^1(I_0 \times \mathbb{R}^n)} + \|u_{< \frac{1}{\eta_2}}\|_{L_t^\infty L_x^{\frac{2n}{n-2}}(I_0 \times \mathbb{R}^n)} \lesssim \eta_2, \quad (6.3)$$

if $c(\eta_2)$ was chosen sufficiently small. In particular, this implies that u_{lo} has small energy

$$\|u_{lo}\|_{L_t^\infty \dot{H}_x^1(I_0 \times \mathbb{R}^n)} + \|u_{lo}\|_{L_t^\infty L_x^{\frac{2n}{n-2}}(I_0 \times \mathbb{R}^n)} \lesssim \eta_2. \quad (6.4)$$

Using the Bernstein estimate and (6.3), one also sees that u_{hi} has small mass

$$\|u_{hi}\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^n)} \lesssim \sum_{1 < N < \frac{1}{\eta_2}} \|u_N\|_{L_t^\infty L_x^2} + \sum_{N \geq \frac{1}{\eta_2}} \|u_N\|_{L_t^\infty L_x^2} \lesssim \eta_2. \quad (6.5)$$

Our goal is to prove (3.17), which is equivalent to

$$\left\| |u_{hi}|^2 \right\|_{L_t^2 \dot{H}_x^{-\frac{n-3}{2}}(I_0 \times \mathbb{R}^n)} \lesssim \eta_1^{\frac{1}{2}}. \quad (6.6)$$

By a standard continuity argument, it suffices to prove (6.6) under the bootstrap hypothesis

$$\left\| |u_{hi}|^2 \right\|_{L_t^2 \dot{H}_x^{-\frac{n-3}{2}}(I_0 \times \mathbb{R}^n)} \lesssim (C_0 \eta_1)^{\frac{1}{2}}. \quad (6.7)$$

for a large constant C_0 depending on energy but not on any of the η 's.

First, let us note that (6.7) and Lemma 5.6 in [35] imply

$$\left\| |\nabla|^{-\frac{n-3}{4}} u_{hi} \right\|_{L_{t,x}^4(I_0 \times \mathbb{R}^n)} \lesssim (C_0 \eta_1)^{\frac{1}{4}}. \quad (6.8)$$

We now use Proposition 6.1 to derive an interaction Morawetz estimate for $\phi = u_{hi}$.

Proposition 6.2. *With the notation and assumptions above, we have*

$$\begin{aligned} & \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{hi}(t, y)|^2 |u_{hi}(t, x)|^2}{|x - y|^3} - |u_{hi}(t, y)|^2 \frac{x - y}{|x - y|} \left(\nabla |\nabla|^{-(n-4)} |u_{hi}|^2 |u_{hi}|^2 \right) (t, x) \, dx dy dt \\ & \leq \eta_2^3 \end{aligned} \quad (6.9)$$

$$+ \eta_2 \iint_{I_0 \times \mathbb{R}^n} |u_{hi} P_{hi}(|\nabla|^{-(n-4)} |u|^2 u - |\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo} - |\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi})|(t, x) dx dt \quad (6.10)$$

$$+ \eta_2 \iint_{I_0 \times \mathbb{R}^n} |u_{hi} P_{lo}(|\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi})|(t, x) dx dt \quad (6.11)$$

$$+ \eta_2 \iint_{I_0 \times \mathbb{R}^n} |u_{hi} P_{hi}(|\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo})|(t, x) dx dt \quad (6.12)$$

$$+ \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} |\nabla|^{-(n-4)} |u_{lo}|^2 |u_{hi}| |\nabla u_{lo}|(t, x) dx dt \quad (6.13)$$

$$+ \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} |\nabla|^{-(n-4)} (|u_{lo}| |u_{hi}|) (|u_{lo}| + |u_{hi}|) |\nabla u_{lo}|(t, x) dx dt \quad (6.14)$$

$$+ \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} |\nabla|^{-(n-4)} |u_{hi}|^2 (|u_{lo}| + |u_{hi}|) |\nabla u_{lo}|(t, x) dx dt \quad (6.15)$$

$$+ \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} \left| \nabla |\nabla|^{-(n-4)} |u_{lo}|^2 (|u_{lo}| |u_{hi}| + |u_{hi}|^2) (t, x) \right| dx dt \quad (6.16)$$

$$+ \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} \left| \nabla |\nabla|^{-(n-4)} (|u_{lo}| |u_{hi}|) (|u_{lo}|^2 + |u_{hi}|^2) (t, x) \right| dx dt \quad (6.17)$$

$$+ \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} \left| \nabla |\nabla|^{-(n-4)} |u_{hi}|^2 (|u_{lo}|^2 + |u_{lo}| |u_{hi}|) (t, x) \right| dx dt \quad (6.18)$$

$$+ \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} |\nabla P_{lo}(|\nabla|^{-(n-4)} |u|^2 u)|(t, x) |u_{hi}|(t, x) \, dx dt \quad (6.19)$$

$$+ \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{hi}(t, y)|^2}{|x - y|} |\nabla|^{-(n-4)} |u_{lo}|^2 (|u_{lo}| |u_{hi}| + |u_{hi}|^2) (t, x) dx dy dt \quad (6.20)$$

$$+ \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{hi}(t, y)|^2}{|x - y|} |\nabla|^{-(n-4)} (|u_{lo}| |u_{hi}|) (|u_{lo}|^2 + |u_{lo}| |u_{hi}| + |u_{hi}|^2) (t, x) dx dy dt \quad (6.21)$$

$$+ \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{hi}(t, y)|^2}{|x - y|} |\nabla|^{-(n-4)} |u_{hi}|^2 (|u_{lo}|^2 + |u_{lo}| |u_{hi}|) (t, x) \, dx dy dt \quad (6.22)$$

$$+ \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{|P_{lo}(|\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi})|(t, x) |u_{hi}|(t, x)}{|x - y|} \, dx dy dt \quad (6.23)$$

Remark 6.1. *By symmetry, we have*

$$- \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{x - y}{|x - y|} \left(\nabla |\nabla|^{-(n-4)} |u_{hi}|^2 |u_{hi}|^2 \right) (t, x) \, dx dy dt \geq 0.$$

Proof: Applying Proposition 6.1 with $\phi = u_{hi}$ and $\mathcal{N} = P_{hi}(|\nabla|^{-(n-4)}|u|^2u)$, we have

$$\begin{aligned}
& \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} (n-1)(n-3) \frac{|u_{hi}(t, y)|^2 |u_{hi}(t, x)|^2}{|x-y|^3} dx dy dt \\
& + 2 \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{x-y}{|x-y|} \{P_{hi}(|\nabla|^{-(n-4)}|u|^2u), u_{hi}\}_p(t, x) dx dy dt \\
& \leq 4 \|u_{hi}\|_{L^\infty L_x^2(I_0 \times \mathbb{R}^n)}^3 \|u_{hi}\|_{L_t^\infty \dot{H}_x^1(I_0 \times \mathbb{R}^n)} \\
& + 4 \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |\{P_{hi}(|\nabla|^{-(n-4)}|u|^2u), u_{hi}\}_m(t, y)| |\nabla u_{hi}(t, x)| |u_{hi}(t, x)| dx dy dt.
\end{aligned}$$

Observe that (6.5) plus the conservation of energy implies

$$\|u_{hi}\|_{L^\infty L_x^2(I_0 \times \mathbb{R}^n)}^3 \|u_{hi}\|_{L_t^\infty \dot{H}_x^1(I_0 \times \mathbb{R}^n)} \leq \eta_2^3,$$

which is the error terms (6.9).

We consider the mass bracket term first. Exploiting cancelation, we write

$$\begin{aligned}
\{P_{hi}(|\nabla|^{-(n-4)}|u|^2u), u_{hi}\}_m &= \{P_{hi}(|\nabla|^{-(n-4)}|u|^2u - |\nabla|^{-(n-4)}|u_{lo}|^2u_{lo} - |\nabla|^{-(n-4)}|u_{hi}|^2u_{hi}), u_{hi}\}_m \\
&\quad - \{P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2u_{hi}), u_{hi}\}_m + \{P_{hi}(|\nabla|^{-(n-4)}|u_{lo}|^2u_{lo}), u_{hi}\}_m.
\end{aligned}$$

Because

$$\int_{\mathbb{R}^n} |\nabla u_{hi}(t, x)| |u_{hi}(t, x)| dx \leq \|u_{hi}\|_{L^\infty L_x^2(I_0 \times \mathbb{R}^n)} \|u_{hi}\|_{L_t^\infty \dot{H}_x^1(I_0 \times \mathbb{R}^n)} \leq \eta_2,$$

we can bound the contribution of the mass bracket term by the following

$$\begin{aligned}
& \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |\{P_{hi}(|\nabla|^{-(n-4)}|u|^2u), u_{hi}\}_m(t, y)| |\nabla u_{hi}(t, x)| |u_{hi}(t, x)| dx dy dt \\
& \leq \eta_2 \iint_{I_0 \times \mathbb{R}^n} |u_{hi}(t, x)| |P_{hi}(|\nabla|^{-(n-4)}|u|^2u - |\nabla|^{-(n-4)}|u_{lo}|^2u_{lo} - |\nabla|^{-(n-4)}|u_{hi}|^2u_{hi})(t, x)| dx dt \\
& \quad + \eta_2 \iint_{I_0 \times \mathbb{R}^n} |u_{hi}(t, x)| (|P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2u_{hi})| + |P_{hi}(|\nabla|^{-(n-4)}|u_{lo}|^2u_{lo})|)(t, x) dx dt
\end{aligned}$$

which are the error terms (6.10), (6.11) and (6.12).

We turn now towards the momentum bracket term and write

$$\begin{aligned}
& \{P_{hi}(|\nabla|^{-(n-4)}|u|^2u), u_{hi}\}_p \\
&= \{|\nabla|^{-(n-4)}|u|^2u, u\}_p - \{|\nabla|^{-(n-4)}|u_{lo}|^2u_{lo}, u_{lo}\}_p \\
&\quad - \{|\nabla|^{-(n-4)}|u|^2u - |\nabla|^{-(n-4)}|u_{lo}|^2u_{lo}, u_{lo}\}_p - \{P_{lo}(|\nabla|^{-(n-4)}|u|^2u), u_{hi}\}_p \\
&= -(\nabla|\nabla|^{-(n-4)}|u|^2|u|^2 - \nabla|\nabla|^{-(n-4)}|u_{lo}|^2|u_{lo}|^2) \\
&\quad + \nabla\emptyset[(|\nabla|^{-(n-4)}|u|^2u - |\nabla|^{-(n-4)}|u_{lo}|^2u_{lo})u_{lo}] + \emptyset[(|\nabla|^{-(n-4)}|u|^2u - |\nabla|^{-(n-4)}|u_{lo}|^2u_{lo})\nabla u_{lo}] \\
&\quad - \{P_{lo}(|\nabla|^{-(n-4)}|u|^2u), u_{hi}\}_p =: I + II + III
\end{aligned}$$

where $\emptyset(X)$ denotes an expression which is schematically of the form X .

To estimate the contribution coming from I , we write

$$\begin{aligned}
& \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{x-y}{|x-y|} \left(\nabla |\nabla|^{-(n-4)} |u|^2 |u|^2 - \nabla |\nabla|^{-(n-4)} |u_{lo}|^2 |u_{lo}|^2 \right) (t, x) \, dx dy dt \\
&= \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{x-y}{|x-y|} \left(\nabla |\nabla|^{-(n-4)} |u_{hi}|^2 |u_{hi}|^2 \right) (t, x) \, dx dy dt \\
&+ \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{x-y}{|x-y|} \nabla |\nabla|^{-(n-4)} \left(|u|^2 |u|^2 - |u_{lo}|^2 |u_{lo}|^2 - |u_{hi}|^2 |u_{hi}|^2 \right) (t, x) \, dx dy dt.
\end{aligned}$$

The first term is the left-hand side term in Proposition 6.2. On the other hand, observing that

$$\| |u_{hi}|^2 \|_{L_t^\infty L_x^1(I_0 \times \mathbb{R}^n)} \lesssim \|u_{hi}\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^n)}^2 \lesssim \eta_2^2, \quad (6.24)$$

we take the absolute values inside the integrals and use (6.24) to obtain

$$\begin{aligned}
& \left| \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{x-y}{|x-y|} \left(\nabla |\nabla|^{-(n-4)} |u|^2 |u|^2 - \nabla |\nabla|^{-(n-4)} |u_{lo}|^2 |u_{lo}|^2 \right. \right. \\
& \quad \left. \left. - \nabla |\nabla|^{-(n-4)} |u_{hi}|^2 |u_{hi}|^2 \right) (t, x) \, dx dy dt \right| \\
& \leq \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} \left| \nabla |\nabla|^{-(n-4)} |u_{lo}|^2 (|u_{lo}| |u_{hi}| + |u_{hi}|^2) (t, x) \right| dx dt \\
& \quad + \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} \left| \nabla |\nabla|^{-(n-4)} (|u_{lo}| |u_{hi}|) (|u_{lo}|^2 + |u_{hi}|^2) (t, x) \right| dx dt \\
& \quad + \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} \left| \nabla |\nabla|^{-(n-4)} |u_{hi}|^2 (|u_{lo}|^2 + |u_{lo}| |u_{hi}|) (t, x) \right| dx dt
\end{aligned}$$

which are the error terms (6.16), (6.17) and (6.18).

To estimate the contribution of II . Integrating by parts for the first term, we obtain

$$\begin{aligned}
& \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{|\nabla|^{-(n-4)} |u|^2 u(t, x) - |\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo}(t, x)| |u_{lo}(t, x)|}{|x-y|} \, dx dy dt \\
& + \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \left| |\nabla|^{-(n-4)} |u|^2 u(t, x) - |\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo}(t, x) \right| |\nabla u_{lo}(t, x)| \, dx dy dt.
\end{aligned}$$

We estimate the error term coming from the first term

$$\begin{aligned}
& \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{|\nabla|^{-(n-4)} |u|^2 u(t, x) - |\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo}(t, x)| |u_{lo}(t, x)|}{|x-y|} \, dx dy dt \\
& \leq \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{hi}(t, y)|^2}{|x-y|} |\nabla|^{-(n-4)} |u_{lo}|^2 |u_{lo}| |u_{hi}|(t, x) \, dx dy dt \\
& \quad + \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{hi}(t, y)|^2}{|x-y|} |\nabla|^{-(n-4)} (|u_{lo}| |u_{hi}|) (|u_{lo}|^2 + |u_{lo}| |u_{hi}|) (t, x) \, dx dy dt \\
& \quad + \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{hi}(t, y)|^2}{|x-y|} |\nabla|^{-(n-4)} |u_{hi}|^2 (|u_{lo}|^2 + |u_{lo}| |u_{hi}|) (t, x) \, dx dy dt
\end{aligned}$$

which are controlled by the error terms of (6.20), (6.21) and (6.22).

We now turn to the contribution of the second term. We take the absolute values inside the integrals and use (6.24) to obtain

$$\begin{aligned}
& \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \left| |\nabla|^{-(n-4)}|u|^2 u - |\nabla|^{-(n-4)}|u_{lo}|^2 u_{lo} \right| |\nabla u_{lo}| \, dx dy dt \\
& \leq \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} \left| |\nabla|^{-(n-4)}|u|^2 u(t, x) - |\nabla|^{-(n-4)}|u_{lo}|^2 u_{lo}(t, x) \right| |\nabla u_{lo}(t, x)| \, dx dt \\
& \leq \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} |\nabla|^{-(n-4)}|u_{lo}|^2 |u_{hi}| |\nabla u_{lo}|(t, x) dx dt \\
& \quad + \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} |\nabla|^{-(n-4)}(|u_{lo}| |u_{hi}|) (|u_{lo}| + |u_{hi}|) |\nabla u_{lo}|(t, x) dx dt \\
& \quad + \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} |\nabla|^{-(n-4)}|u_{hi}|^2 (|u_{lo}| + |u_{hi}|) |\nabla u_{lo}|(t, x) \, dx dt
\end{aligned}$$

which are the error terms of (6.13), (6.14) and (6.15).

We consider next the contribution of *III* to the momentum bracket term.

When the derivative falls on $P_{lo}(|\nabla|^{-(n-4)}|u|^2 u)$, we take the absolute values inside the integrals and use (6.24) to estimate this contribution by

$$\begin{aligned}
& \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 |\nabla P_{lo}(|\nabla|^{-(n-4)}|u|^2 u)|(t, x) |u_{hi}|(t, x) \, dx dy dt \\
& \leq \eta_2^2 \iint_{I_0 \times \mathbb{R}^n} |\nabla P_{lo}(|\nabla|^{-(n-4)}|u|^2 u)|(t, x) |u_{hi}|(t, x) \, dx dt,
\end{aligned}$$

which is the error term (6.19).

When the derivative falls on u_{hi} , we first integrate by parts and then take the absolute values inside the integrals to obtain,

$$\begin{aligned}
& \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 |\nabla P_{lo}(|\nabla|^{-(n-4)}|u|^2 u)|(t, x) |u_{hi}|(t, x) \, dx dy dt \\
& + \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{|P_{lo}(|\nabla|^{-(n-4)}|u|^2 u)|(t, x) |u_{hi}|(t, x)}{|x - y|} \, dx dy dt \\
& \leq \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 |\nabla P_{lo}(|\nabla|^{-(n-4)}|u|^2 u)|(t, x) |u_{hi}|(t, x) \, dx dy dt \\
& + \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{|P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{hi})|(t, x) |u_{hi}|(t, x)}{|x - y|} \, dx dy dt \\
& + \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{|P_{lo}(|\nabla|^{-(n-4)}|u|^2 u - |\nabla|^{-(n-4)}|u_{hi}|^2 u_{hi})|(t, x) |u_{hi}|(t, x)}{|x - y|} \, dx dy dt.
\end{aligned}$$

The first term on the right-hand side of the above inequality is controlled by (6.19). The second term is

controlled by (6.23). The third term is estimated by

$$\begin{aligned}
& \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} |u_{hi}(t, y)|^2 \frac{|P_{lo}(|\nabla|^{-(n-4)}|u|^2 u - |\nabla|^{-(n-4)}|u_{hi}|^2 u_{hi})|(t, x)|u_{hi}|(t, x)}{|x - y|} dx dy dt \\
& \leq \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{hi}(t, y)|^2}{|x - y|} |\nabla|^{-(n-4)}|u_{lo}|^2 (|u_{lo}||u_{hi}| + |u_{hi}|^2)(t, x) dx dy dt \\
& \quad + \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{hi}(t, y)|^2}{|x - y|} |\nabla|^{-(n-4)}(|u_{lo}| - |u_{hi}|)(|u_{lo}||u_{hi}| + |u_{hi}|^2)(t, x) dx dy dt \\
& \quad + \iiint_{I_0 \times \mathbb{R}^n \times \mathbb{R}^n} \frac{|u_{hi}(t, y)|^2}{|x - y|} |\nabla|^{-(n-4)}|u_{hi}|^2 (|u_{lo}||u_{hi}|)(t, x) dx dy dt
\end{aligned}$$

which are controlled by (6.20), (6.21) and (6.22).

6.3 Interaction Morawetz: Strichartz control

The purpose of this section is to obtain estimates on the low and high-frequency parts of u , which we will use to bound the error terms in Proposition 6.2.

Proposition 6.3 (Strichartz control on low and high frequencies). *There exists a constant C_1 possibly depending on the energy, but not on any of the η 's, such that*

$$\|u_{lo}\|_{\dot{S}^1(I_0 \times \mathbb{R}^n)} \leq C_1(C_0\eta_1)^{\frac{1}{2}}, \quad (6.25)$$

$$\|u_{hi}\|_{L_t^2 L_x^{\frac{2n}{n-2}}(I_0 \times \mathbb{R}^n)} \leq C_1(C_0\eta_1)^{\frac{1}{2}}. \quad (6.26)$$

Proof: To prove this Proposition, we will use a bootstrap argument. Fix $t_0 := \inf I_0$ and let Ω_1 be the set of all times $T \in I_0$ such that (6.25) and (6.26) hold on $[t_0, T]$.

Define also Ω_2 to be the set of all times $T \in I_0$ such that

$$\|u_{lo}\|_{\dot{S}^1([t_0, T] \times \mathbb{R}^n)} \leq 2C_1(C_0\eta_1)^{\frac{1}{2}} \quad (6.27)$$

$$\|u_{hi}\|_{L_t^2 L_x^{\frac{2n}{n-2}}([t_0, T] \times \mathbb{R}^n)} \leq 2C_1(C_0\eta_1)^{\frac{1}{2}} \quad (6.28)$$

hold.

In order to run a bootstrap argument successfully, we need to check four things:

1. First, we need to see that $t_0 \in \Omega_1$; this follows immediately from the definition of u_{lo} and u_{hi} at time $t = t_0$, provided C_1 is sufficiently large.
2. Next, we need Ω_1 to be closed; this follows from the definition of Ω_1 and Fatou's lemma.
3. Third, we need to prove that if $T \in \Omega_1$, then there exists a small neighborhood of T contained in Ω_2 . This property follows from the dominated convergence theorem and the fact that u_{lo} is not only in $\dot{S}^1([t_0, T] \times \mathbb{R}^n)$, but also in $C_t^0 \dot{H}_x^1([t_0, T] \times \mathbb{R}^n)$ because of the smoothing effect of the free propagator.
4. The last step one needs to check is that $\Omega_2 \subset \Omega_1$ and this is what we will focus on for the rest of the proof of Proposition 6.3. Now, we fix $T \in \Omega_2$. Throughout the rest of the proof, all spacetime norms will be on $[t_0, T] \times \mathbb{R}^n$.

We first consider the low frequencies. By the Strichartz estimate, we have

$$\begin{aligned}
\|u_{lo}\|_{\dot{S}^1} &\lesssim \|\nabla u_{lo}(t_0)\|_{L_t^\infty L_x^2} \\
&+ \|\nabla P_{lo}(|\nabla|^{-(n-4)}|u_{lo}|^2 u_{lo})\|_{L^2 L^{\frac{2n}{n+2}}} + \|\nabla P_{lo}(|\nabla|^{-(n-4)}|u_{lo}|^2 u_{hi})\|_{L^2 L^{\frac{2n}{n+2}}} \\
&+ \|\nabla P_{lo}(|\nabla|^{-(n-4)}\operatorname{Re}(\bar{u}_{hi} u_{lo}) u_{lo})\|_{L^2 L^{\frac{2n}{n+2}}} + \|\nabla P_{lo}(|\nabla|^{-(n-4)}\operatorname{Re}(\bar{u}_{hi} u_{lo}) u_{hi})\|_{L^2 L^{\frac{2n}{n+2}}} \\
&+ \|\nabla P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{lo})\|_{L^2 L^{\frac{2n}{n+2}}} + \|\nabla P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{hi})\|_{L^{\frac{4}{3}} L^{\frac{2n}{n+1}}}.
\end{aligned}$$

By (6.4), we have

$$\|\nabla u_{lo}(t_0)\|_{L_x^2} \lesssim \eta_2.$$

By the Hölder inequality, Hardy-Littlewood-Sobolev inequality and (6.4), we get

$$\|\nabla P_{lo}(|\nabla|^{-(n-4)}|u_{lo}|^2 u_{lo})\|_{L^2 L^{\frac{2n}{n+2}}} \lesssim \|\nabla u_{lo}\|_{L^2 L^{\frac{2n}{n-2}}} \|u_{lo}\|_{L^\infty L^{\frac{2n}{n-2}}}^2 \lesssim 2C_1(C_0\eta_1)^{\frac{1}{2}}\eta_2^2.$$

Similarly, by the Bernstein estimate, Hölder inequality, Hardy-Littlewood-Sobolev inequality, (6.4) and (6.5), we get

$$\begin{aligned}
\|\nabla P_{lo}(|\nabla|^{-(n-4)}|u_{lo}|^2 u_{hi})\|_{L^2 L^{\frac{2n}{n+2}}} &\lesssim \|P_{lo}(|\nabla|^{-(n-4)}|u_{lo}|^2 u_{hi})\|_{L^2 L^{\frac{2n}{n+2}}} \\
&\lesssim \|u_{lo}\|_{L^4 L^{\frac{2n}{n-3}}}^2 \|u_{hi}\|_{L^\infty L^2} \lesssim (2C_1)^2 C_0 \eta_1 \eta_2, \\
\|\nabla P_{lo}(|\nabla|^{-(n-4)}\operatorname{Re}(\bar{u}_{hi} u_{lo}) u_{lo})\|_{L^2 L^{\frac{2n}{n+2}}} &\lesssim (2C_1)^2 C_0 \eta_1 \eta_2 \\
\|\nabla P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{lo})\|_{L^2 L^{\frac{2n}{n+2}}} &\lesssim \|P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{lo})\|_{L^2 L^{\frac{2n}{n+2}}} \\
&\lesssim \|P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{lo})\|_{L^2 L^{\frac{2n}{n+4}}} \\
&\lesssim \|u_{lo}\|_{L^2 L^{\frac{2n}{n-4}}} \|u_{hi}\|_{L^\infty L^2}^2 \lesssim 2C_1(C_0\eta_1)^{\frac{1}{2}}\eta_2^2, \\
\|\nabla P_{lo}(|\nabla|^{-(n-4)}\operatorname{Re}(\bar{u}_{hi} u_{lo}) u_{hi})\|_{L^2 L^{\frac{2n}{n+2}}} &\lesssim 2C_1(C_0\eta_1)^{\frac{1}{2}}\eta_2^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{hi})\|_{L^{\frac{4}{3}} L^{\frac{2n}{n+1}}} \\
&\lesssim \|P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{hi})\|_{L^{\frac{4}{3}} L^{\frac{2n}{n+1}}} \lesssim \|P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{hi})\|_{L^{\frac{4}{3}} L^{\frac{2n}{n+5}}} \\
&\lesssim \|u_{hi}\|_{L^4 L^{\frac{2n}{n-1}}}^3 \lesssim \|u_{hi}\|_{L^2 L^{\frac{2n}{n-2}}}^{\frac{3}{2}} \|u_{hi}\|_{L^\infty L^2}^{\frac{3}{2}} \lesssim (2C_1(C_0\eta_1)^{\frac{1}{2}})^{\frac{3}{2}} \eta_2^{\frac{3}{2}}.
\end{aligned}$$

Combining the above estimates, we get

$$\|u_{lo}\|_{\dot{S}^1} \lesssim \eta_2 + 2C_1(C_0\eta_1)^{\frac{1}{2}}\eta_2^2 + (2C_1)^2 C_0 \eta_1 \eta_2 + (2C_1(C_0\eta_1)^{\frac{1}{2}})^{\frac{3}{2}} \eta_2^{\frac{3}{2}} \leq C_1(C_0\eta_1)^{\frac{1}{2}},$$

provided we choose η_2 sufficiently small.

We turn now toward the high frequencies of u . By the Strichartz estimate,

$$\begin{aligned}
&\|u_{hi}\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \\
&\lesssim \|u_{hi}(t_0)\|_{L_x^2} \\
&+ \|P_{hi}(|\nabla|^{-(n-4)}|u_{lo}|^2 u_{lo})\|_{L_t^2 L_x^{\frac{2n}{n+2}}} + \|P_{hi}(|\nabla|^{-(n-4)}|u_{lo}|^2 u_{hi})\|_{L_t^2 L_x^{\frac{2n}{n+2}}} \\
&+ \|P_{hi}(|\nabla|^{-(n-4)}\operatorname{Re}(\bar{u}_{hi} u_{lo}) u_{lo})\|_{L_t^2 L_x^{\frac{2n}{n+2}}} + \|P_{hi}(|\nabla|^{-(n-4)}\operatorname{Re}(\bar{u}_{hi} u_{lo}) u_{hi})\|_{L_t^2 L_x^{\frac{2n}{n+2}}} \\
&+ \|P_{hi}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{lo})\|_{L_t^2 L_x^{\frac{2n}{n+2}}} + \|P_{hi}(|\nabla|^{-(n-4)}|u_{hi}|^2 u_{hi})\|_{L_t^2 L_x^{\frac{2n}{n+2}}}.
\end{aligned}$$

By (6.5), we have

$$\|u_{hi}(t_0)\|_{L_x^2} \lesssim \eta_2.$$

By the Bernstein estimate, Hölder inequality, Hardy-Littlewood-Sobolev inequality, (6.4), (6.5) and (6.8), we get

$$\begin{aligned} \|P_{hi}(|\nabla|^{-(n-4)}|u_{lo}|^2u_{lo})\|_{L_t^2L_x^{\frac{2n}{n+2}}} &\lesssim \|\nabla P_{hi}(|\nabla|^{-(n-4)}|u_{lo}|^2u_{lo})\|_{L^2L^{\frac{2n}{n+2}}} \\ &\lesssim \|\nabla u_{lo}\|_{L^2L^{\frac{2n}{n-2}}} \|u_{lo}\|_{L^\infty L^{\frac{2n}{n-2}}}^2 \lesssim 2C_1(C_0\eta_1)^{\frac{1}{2}}\eta_2^2, \\ \|P_{hi}(|\nabla|^{-(n-4)}|u_{lo}|^2u_{hi})\|_{L_t^2L_x^{\frac{2n}{n+2}}} &\lesssim \|u_{lo}\|_{L^4L^{\frac{2n}{n-3}}}^2 \|u_{hi}\|_{L^\infty L^2} \lesssim (2C_1)^2C_0\eta_1\eta_2, \\ \|P_{hi}(|\nabla|^{-(n-4)}\operatorname{Re}(\overline{u_{hi}}u_{lo})u_{lo})\|_{L_t^2L_x^{\frac{2n}{n+2}}} &\lesssim (2C_1)^2C_0\eta_1\eta_2, \\ \|P_{hi}(|\nabla|^{-(n-4)}|u_{hi}|^2u_{lo})\|_{L_t^2L_x^{\frac{2n}{n+2}}} &\lesssim \|P_{hi}(|\nabla|^{-(n-4)}|u_{hi}|^2u_{lo})\|_{L_t^2L_x^{\frac{2n}{n+3}}} \\ &\lesssim \|u_{hi}\|_{L^4L^{\frac{2n}{n-1}}}^2 \|u_{lo}\|_{L^\infty L^{\frac{2n}{n-3}}} \\ &\lesssim \|u_{hi}\|_{L^4L^{\frac{2n}{n-1}}}^2 \|u_{lo}\|_{L^\infty L^{\frac{2n}{n-2}}} \lesssim 2C_1(C_0\eta_1)^{\frac{1}{2}}\eta_2^{\frac{1}{2}}\eta_2, \\ \|P_{hi}(|\nabla|^{-(n-4)}\operatorname{Re}(\overline{u_{hi}}u_{lo})u_{hi})\|_{L_t^2L_x^{\frac{2n}{n+2}}} &\lesssim 2C_1(C_0\eta_1)^{\frac{1}{2}}\eta_2^{\frac{1}{2}}\eta_2, \end{aligned}$$

and

$$\begin{aligned} \|P_{hi}(|\nabla|^{-(n-4)}|u_{hi}|^2u_{hi})\|_{L_t^2L_x^{\frac{2n}{n+2}}} &\lesssim \|P_{hi}(|\nabla|^{-(n-4)}|u_{hi}|^2u_{hi})\|_{L_t^2L_x^{\frac{2n}{n+3}}} \lesssim \|u_{hi}\|_{L^6L^{\frac{6n}{3n-5}}}^3 \\ &\lesssim \| |\nabla|^{-\frac{n-3}{4}}u_{hi}\|_{L_{t,x}^4}^{\frac{2}{3}} \|\nabla u_{hi}\|_{L^\infty L^2}^{\frac{4}{3}} \|u_{hi}\|_{L^3L^{\frac{6n}{3n-4}}} \\ &\lesssim (C_0\eta_1)^{\frac{1}{4}\cdot\frac{2}{3}} (2C_1(C_0\eta_1)^{\frac{1}{2}})^{\frac{2}{3}} = (2C_1)^{\frac{2}{3}}(C_0\eta_1)^{\frac{1}{2}}. \end{aligned}$$

Combining the above estimates, we obtain

$$\begin{aligned} \|u_{hi}\|_{L_t^2L_x^{\frac{2n}{n-2}}} &\lesssim \eta_2 + 2C_1(C_0\eta_1)^{\frac{1}{2}}\eta_2^2 + (2C_1)^2C_0\eta_1\eta_2 + 2C_1(C_0\eta_1)^{\frac{1}{2}}\eta_2^{\frac{1}{2}}\eta_2 + (2C_1)^{\frac{2}{3}}(C_0\eta_1)^{\frac{1}{2}} \\ &\leq C_1(C_0\eta_1)^{\frac{1}{2}}, \end{aligned}$$

provided C_1 sufficiently large and η_2 sufficiently small.

Remark 6.2. Interpolating between (6.4) and (6.25), for any Schrödinger sharp admissible pair (q, r) we obtain

$$\|\nabla u_{lo}\|_{L^qL^r(I_0 \times R^n)} \lesssim C_1^{\frac{2}{q}}(C_0\eta_1)^{\frac{1}{q}}\eta_2^{1-\frac{2}{q}} \lesssim (C_0\eta_1)^{\frac{1}{q}}. \quad (6.29)$$

Similarly, interpolating between (6.5), (6.26) and the boundness of the energy, for any Schrödinger sharp admissible pair (q, r) , we get

$$\|u_{hi}\|_{L^qL^r(I_0 \times R^n)} \lesssim C_1^{\frac{2}{q}}(C_0\eta_1)^{\frac{1}{q}}\eta_2^{1-\frac{2}{q}} \lesssim (C_0\eta_1)^{\frac{1}{q}}. \quad (6.30)$$

6.4 Interaction Morawetz: Error estimates

In this section, we use the control on u_{lo} and u_{hi} in Proposition 6.3 to bound the error terms on the right-hand side of Proposition 6.2. Throughout the rest of the section all spacetime norms will be on $I_0 \times R^n$.

Consider (6.10), by the Bernstein estimate, Hölder inequality, Sobolev embedding, Proposition 6.3, and Remark 6.2, we have

$$\begin{aligned}
(6.10) &\lesssim \eta_2 \left(\| |u_{hi}| |\nabla|^{-(n-4)} |u_{hi}|^2 u_{lo} \|_{L^1 L^1} + \| |u_{hi}| |\nabla|^{-(n-4)} |u_{lo}|^2 u_{hi} \|_{L^1 L^1} \right. \\
&\quad \left. + \| |u_{hi}| |\nabla|^{-(n-4)} \operatorname{Re}(\bar{u}_{hi} u_{lo}) u_{hi} \|_{L^1 L^1} + \| |u_{hi}| |\nabla|^{-(n-4)} \operatorname{Re}(\bar{u}_{hi} u_{lo}) u_{lo} \|_{L^1 L^1} \right) \\
&\lesssim \eta_2 \left(\| |u_{hi}| \|_{L^4 L^{\frac{2n}{n-1}}}^2 \| |u_{hi}| \|_{L^\infty L^{\frac{2n}{n-2}}} \| |u_{lo}| \|_{L^2 L^{\frac{2n}{n-4}}} + \| |u_{hi}| \|_{L^4 L^{\frac{2n}{n-1}}}^2 \| |u_{lo}| \|_{L^4 L^{\frac{2n}{n-3}}}^2 \right) \\
&\lesssim \eta_2 \left((C_0 \eta_1)^{\frac{1}{2}} (C_0 \eta_1)^{\frac{1}{2}} + (C_0 \eta_1)^{\frac{1}{2}} (C_0 \eta_1)^{\frac{1}{2}} \right) \lesssim \eta_2 C_0 \eta_1.
\end{aligned}$$

We now move on to (6.11). Using the Bernstein estimate, Proposition 6.3, and Remark 6.2, we get

$$\begin{aligned}
(6.11) &\lesssim \eta_2 \| |u_{hi}| P_{lo}(|\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi}) \|_{L^1 L^1} \\
&\lesssim \eta_2 \| |u_{hi}| \|_{L^3 L^{\frac{6n}{3n-4}}} \| P_{lo}(|\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi}) \|_{L^{\frac{3}{2}} L^{\frac{6n}{3n+4}}} \\
&\lesssim \eta_2 \| |u_{hi}| \|_{L^3 L^{\frac{6n}{3n-4}}} \| P_{lo}(|\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi}) \|_{L^{\frac{3}{2}} L^{\frac{6n}{3n+10}}} \\
&\lesssim \eta_2 \| |u_{hi}| \|_{L^3 L^{\frac{6n}{3n-4}}}^3 \| |u_{hi}| \|_{L^\infty L^{\frac{2n}{n-2}}} \lesssim \eta_2 C_0 \eta_1.
\end{aligned}$$

We next estimate (6.12). By the Bernstein estimate, Sobolev embedding, Proposition 6.3, and Remark 6.2, we estimate

$$\begin{aligned}
(6.12) &\lesssim \eta_2 \| |u_{hi}| P_{hi}(|\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo}) \|_{L^1 L^1} \\
&\lesssim \eta_2 \| |u_{hi}| \|_{L^4 L^{\frac{2n}{n-1}}} \| P_{hi}(|\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo}) \|_{L^{\frac{4}{3}} L^{\frac{2n}{n+1}}} \\
&\lesssim \eta_2 \| |u_{hi}| \|_{L^4 L^{\frac{2n}{n-1}}} \| \nabla P_{hi}(|\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo}) \|_{L^{\frac{4}{3}} L^{\frac{2n}{n+1}}} \\
&\lesssim \eta_2 \| |u_{hi}| \|_{L^4 L^{\frac{2n}{n-1}}} \| \nabla u_{lo} \|_{L^4 L^{\frac{2n}{n-1}}} \| |u_{lo}| \|_{L^4 L^{\frac{2n}{n-3}}}^2 \lesssim \eta_2 C_0 \eta_1.
\end{aligned}$$

We now turn toward (6.13) – (6.15), and use the Hölder, Sobolev embedding, Proposition 6.3, and Remark 6.2 to obtain

$$\begin{aligned}
(6.13) &\lesssim \eta_2^2 \| |\nabla|^{-(n-4)} |u_{lo}|^2 |u_{hi}| |\nabla u_{lo}| \|_{L^1 L^1} \\
&\lesssim \eta_2^2 \| |u_{hi}| \|_{L^4 L^{\frac{2n}{n-1}}} \| \nabla u_{lo} \|_{L^4 L^{\frac{2n}{n-1}}} \| |u_{lo}| \|_{L^4 L^{\frac{2n}{n-3}}}^2 \lesssim \eta_2^2 C_0 \eta_1, \\
(6.14) &\lesssim \eta_2^2 \| |\nabla|^{-(n-4)} |u_{lo}| u_{hi} (|u_{lo}| + |u_{hi}|) |\nabla u_{lo}| \|_{L^1 L^1} \\
&\lesssim \eta_2^2 \| |u_{hi}| \|_{L^4 L^{\frac{2n}{n-1}}} \| \nabla u_{lo} \|_{L^4 L^{\frac{2n}{n-1}}} \| |u_{lo}| \|_{L^4 L^{\frac{2n}{n-3}}}^2 \\
&\quad + \eta_2^2 \| |u_{hi}| \|_{L^3 L^{\frac{6n}{3n-4}}} \| |u_{hi}| \|_{L^\infty L^{\frac{2n}{n-2}}} \| |u_{lo}| \|_{L^3 L^{\frac{6n}{3n-10}}} \| \nabla u_{lo} \|_{L^3 L^{\frac{6n}{3n-4}}} \lesssim \eta_2^2 C_0 \eta_1, \\
(6.15) &\lesssim \eta_2^2 \| |\nabla|^{-(n-4)} |u_{hi}|^2 (|u_{lo}| + |u_{hi}|) |\nabla u_{lo}| \|_{L^1 L^1} \\
&\lesssim \eta_2^2 \| |u_{hi}| \|_{L^3 L^{\frac{6n}{3n-4}}} \| |u_{hi}| \|_{L^\infty L^{\frac{2n}{n-2}}} \| |u_{lo}| \|_{L^3 L^{\frac{6n}{3n-10}}} \| \nabla u_{lo} \|_{L^3 L^{\frac{6n}{3n-4}}} \\
&\quad + \eta_2^2 \| |u_{hi}| \|_{L^3 L^{\frac{6n}{3n-4}}}^2 \| |u_{hi}| \|_{L^\infty L^{\frac{2n}{n-2}}} \| \nabla u_{lo} \|_{L^3 L^{\frac{6n}{3n-10}}} \lesssim \eta_2^2 C_0 \eta_1,
\end{aligned}$$

where in the last inequality we use the fact that

$$\| \nabla u_{lo} \|_{L^3 L^{\frac{6n}{3n-10}}} \lesssim \| |u_{lo}| \|_{L^3 L^{\frac{6n}{3n-10}}} \lesssim (C_0 \eta_1)^{\frac{1}{3}}.$$

We next consider (6.16) – (6.18), and use the Hölder, Sobolev embedding, Proposition 6.3, and Remark

6.2 to obtain

$$\begin{aligned}
(6.16) &\lesssim \eta_2^2 \|\nabla |\nabla|^{-(n-4)} |u_{lo}|^2 (|u_{lo}| |u_{hi}| + |u_{hi}|^2)\|_{L^1 L^1} \\
&\lesssim \eta_2^2 \|u_{hi}\|_{L^4 L^{\frac{2n}{n-1}}} \|u_{lo}\|_{L^4 L^{\frac{2n}{n-3}}}^3 + \eta_2^2 \|u_{hi}\|_{L^3 L^{\frac{6n}{3n-4}}} \|u_{hi}\|_{L^\infty L^{\frac{2n}{n-2}}} \|u_{lo}\|_{L^3 L^{\frac{6n}{3n-10}}}^2 \lesssim \eta_2^2 C_0 \eta_1, \\
(6.17) &\lesssim \eta_2^2 \|\nabla |\nabla|^{-(n-4)} (|u_{lo}| |u_{hi}|) (|u_{lo}|^2 + |u_{hi}|^2)\|_{L^1 L^1} \\
&\lesssim \eta_2^2 \|u_{hi}\|_{L^4 L^{\frac{2n}{n-1}}} \|u_{lo}\|_{L^4 L^{\frac{2n}{n-3}}}^3 + \eta_2^2 \|u_{hi}\|_{L^3 L^{\frac{6n}{3n-4}}}^2 \|u_{hi}\|_{L^\infty L^{\frac{2n}{n-2}}} \|u_{lo}\|_{L^3 L^{\frac{6n}{3n-10}}} \lesssim \eta_2^2 C_0 \eta_1, \\
(6.18) &\lesssim \eta_2^2 \|\nabla |\nabla|^{-(n-4)} |u_{hi}|^2 (|u_{lo}|^2 + |u_{lo}| |u_{hi}|)\|_{L^1 L^1} \\
&\lesssim \eta_2^2 \|u_{hi}\|_{L^3 L^{\frac{6n}{3n-4}}} \|u_{hi}\|_{L^\infty L^{\frac{2n}{n-2}}} \|u_{lo}\|_{L^3 L^{\frac{6n}{3n-10}}}^2 + \eta_2^2 \|u_{hi}\|_{L^3 L^{\frac{6n}{3n-4}}}^{\frac{3}{2}} \|u_{hi}\|_{L^\infty L^{\frac{2n}{n-2}}}^{\frac{3}{2}} \|u_{lo}\|_{L^2 L^\nu} \\
&\lesssim \eta_2^2 C_0 \eta_1,
\end{aligned}$$

where $\nu = \infty$ for $n = 5$ and $\nu = \frac{2n}{n-5}$ for $n \geq 6$ and we use the fact that

$$\|u_{lo}\|_{L^2 L^\nu} \lesssim \|u_{lo}\|_{L^2 L^{\frac{2n}{n-4}}} \lesssim (C_0 \eta_1)^{\frac{1}{2}}$$

in the last inequality.

Now we turn toward (6.19). By the triangle inequality and the similar estimates as (6.10), (6.11) and (6.12), we get

$$\begin{aligned}
(6.19) &\lesssim \eta_2^2 \|u_{hi} \nabla P_{lo} (|\nabla|^{-(n-4)} |u|^2 u)\|_{L^1 L^1} \\
&\lesssim \eta_2^2 \|u_{hi} \nabla P_{lo} (|\nabla|^{-(n-4)} (|u|^2 u - |u_{lo}|^2 u_{lo} - |u_{hi}|^2 u_{hi}))\|_{L^1 L^1} \\
&\quad + \eta_2^2 \|u_{hi} \nabla P_{lo} (|\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo})\|_{L^1 L^1} \\
&\quad + \eta_2^2 \|u_{hi} \nabla P_{lo} (|\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi})\|_{L^1 L^1} \\
&\lesssim \eta_2^2 \|u_{hi}\|_{L^4 L^{\frac{2n}{n-1}}} \| |\nabla|^{-(n-4)} (|u|^2 u - |u_{lo}|^2 u_{lo} - |u_{hi}|^2 u_{hi}) \|_{L^{\frac{4}{3}} L^{\frac{2n}{n+1}}} \\
&\quad + \eta_2^2 \|u_{hi}\|_{L^4 L^{\frac{2n}{n-1}}} \|\nabla P_{lo} (|\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo})\|_{L^{\frac{4}{3}} L^{\frac{2n}{n+1}}} \\
&\quad + \eta_2^2 \|u_{hi}\|_{L^3 L^{\frac{6n}{3n-4}}} \|P_{lo} (|\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi})\|_{L^{\frac{3}{2}} L^{\frac{6n}{3n+4}}} \\
&\lesssim \eta_2^2 C_0 \eta_1.
\end{aligned}$$

We turn now to the error terms (6.20) through (6.23). We notice that they are of the form $\langle |u_{hi}|^2 * \frac{1}{|x|}, f \rangle$ where

$$f = \begin{cases} |\nabla|^{-(n-4)} |u_{lo}|^2 (|u_{lo}| |u_{hi}| + |u_{hi}|^2) & \text{in (6.20),} \\ |\nabla|^{-(n-4)} (|u_{lo}| |u_{hi}|) (|u_{lo}|^2 + |u_{lo}| |u_{hi}| + |u_{hi}|^2) & \text{in (6.21),} \\ |\nabla|^{-(n-4)} |u_{hi}|^2 (|u_{lo}|^2 + |u_{lo}| |u_{hi}|) & \text{in (6.22),} \\ |u_{hi} P_{lo} (|\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi})| & \text{in (6.23).} \end{cases}$$

Let us first note that as $u_{hi} \in L^3 L^{\frac{6n}{3n-4}}$ and $u_{hi} \in L^\infty L^2$, we also have $|u_{hi}|^2 \in L^3 L^{\frac{3n}{3n-2}}$. Therefore, by the Hardy-Littlewood-Sobolev inequality, we have $|u_{hi}|^2 * \frac{1}{|x|} \in L^3 L^{3n}$ and

$$\begin{aligned}
\langle |u_{hi}|^2 * \frac{1}{|x|}, f \rangle &\lesssim \| |u_{hi}|^2 * \frac{1}{|x|} \|_{L^3 L^{3n}} \|f\|_{L^{\frac{3}{2}} L^{\frac{3n}{3n-1}}} \\
&\lesssim \|u_{hi}\|_{L^3 L^{\frac{6n}{3n-4}}} \|u_{hi}\|_{L^\infty L^2} \|f\|_{L^{\frac{3}{2}} L^{\frac{3n}{3n-1}}} \lesssim (C_0 \eta_1)^{\frac{1}{3}} \eta_2 \|f\|_{L^{\frac{3}{2}} L^{\frac{3n}{3n-1}}}.
\end{aligned}$$

Consider the case of (6.20), that is $f = |\nabla|^{-(n-4)}|u_{lo}|^2(|u_{lo}||u_{hi}| + |u_{hi}|^2)$. By the Hölder inequality, Hardy-Littlewood-Sobolev inequality, Proposition 6.3, and Remark 6.2, we estimate

$$\begin{aligned}
(6.20) &\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|\left|\nabla\right|^{-(n-4)}|u_{lo}|^2(|u_{lo}||u_{hi}| + |u_{hi}|^2)\right\|_{L^{\frac{3}{2}}L^{\frac{3n}{3n-1}}} \\
&\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|u_{lo}\right\|_{L^6L^{\frac{6n}{3n-8}}}^2\left\|u_{hi}\right\|_{L^3L^{\frac{6n}{3n-4}}}\left(\left\|u_{lo}\right\|_{L^\infty L^{\frac{2n}{n-2}}} + \left\|u_{hi}\right\|_{L^\infty L^{\frac{2n}{n-2}}}\right) \\
&\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2(C_0\eta_1)^{\frac{1}{3}}(C_0\eta_1)^{\frac{1}{3}}\lesssim C_0\eta_1\eta_2.
\end{aligned}$$

Consider next the error term (6.21), that is, $f = |\nabla|^{-(n-4)}(|u_{lo}||u_{hi}|)(|u_{lo}|^2 + |u_{lo}||u_{hi}| + |u_{hi}|^2)$. By the Hölder inequality, Hardy-Littlewood-Sobolev inequality, Proposition 6.3, and Remark 6.2, we have

$$\begin{aligned}
(6.21) &\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|\left|\nabla\right|^{-(n-4)}(|u_{lo}||u_{hi}|)(|u_{lo}|^2 + |u_{lo}||u_{hi}| + |u_{hi}|^2)\right\|_{L^{\frac{3}{2}}L^{\frac{3n}{3n-1}}} \\
&\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|u_{lo}\right\|_{L^3L^{\frac{6n}{3n-10}}}^2\left\|u_{lo}\right\|_{L^\infty L^{\frac{2n}{n-2}}}^2\left\|u_{hi}\right\|_{L^3L^{\frac{6n}{3n-4}}} \\
&\quad + (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|u_{lo}\right\|_{L^3L^{\frac{6n}{3n-10}}}\left\|u_{lo}\right\|_{L^\infty L^{\frac{2n}{n-2}}}\left\|u_{hi}\right\|_{L^\infty L^{\frac{2n}{n-2}}}\left\|u_{hi}\right\|_{L^3L^{\frac{6n}{3n-4}}} \\
&\quad + (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|u_{lo}\right\|_{L^3L^{\frac{6n}{3n-10}}}\left\|u_{hi}\right\|_{L^\infty L^{\frac{2n}{n-2}}}^2\left\|u_{hi}\right\|_{L^3L^{\frac{6n}{3n-4}}} \lesssim C_0\eta_1\eta_2.
\end{aligned}$$

Similarly, we can estimate the error term (6.22), that is,

$$\begin{aligned}
(6.22) &\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|\left|\nabla\right|^{-(n-4)}|u_{hi}|^2(|u_{lo}|^2 + |u_{lo}||u_{hi}|)\right\|_{L^{\frac{3}{2}}L^{\frac{3n}{3n-1}}} \\
&\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|u_{lo}\right\|_{L^3L^{\frac{6n}{3n-10}}}\left\|u_{lo}\right\|_{L^\infty L^{\frac{2n}{n-2}}}\left\|u_{hi}\right\|_{L^\infty L^{\frac{2n}{n-2}}}\left\|u_{hi}\right\|_{L^3L^{\frac{6n}{3n-4}}} \\
&\quad + (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|u_{lo}\right\|_{L^3L^{\frac{6n}{3n-10}}}\left\|u_{hi}\right\|_{L^\infty L^{\frac{2n}{n-2}}}^2\left\|u_{hi}\right\|_{L^3L^{\frac{6n}{3n-4}}} \lesssim C_0\eta_1\eta_2.
\end{aligned}$$

The last error term left to estimate is (6.23). Using the Bernstein estimate, Proposition 6.3, and Remark 6.2, we obtain

$$\begin{aligned}
(6.23) &\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|u_{hi}P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2u_{hi})\right\|_{L^{\frac{3}{2}}L^{\frac{3n}{3n-1}}} \\
&\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|u_{hi}\right\|_{L^3L^{\frac{6n}{3n-4}}}\left\|P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2u_{hi})\right\|_{L^3L^{\frac{6n}{3n+2}}} \\
&\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|u_{hi}\right\|_{L^3L^{\frac{6n}{3n-4}}}\left\|P_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2u_{hi})\right\|_{L^3L^{\frac{6n}{3n+8}}} \\
&\lesssim (C_0\eta_1)^{\frac{1}{3}}\eta_2\left\|u_{hi}\right\|_{L^3L^{\frac{6n}{3n-4}}}^2\left\|u_{hi}\right\|_{L^\infty L^{\frac{2n}{n-2}}}^2 \lesssim C_0\eta_1\eta_2.
\end{aligned}$$

Collecting all the above estimates, we obtain that all the error terms on the right-hand side of Proposition 6.2 are controlled by η_1 . Upon rescaling, this concludes the proof of Proposition 3.17 in all dimensions $n \geq 5$.

7 Preventing energy evacuation

We now prove Proposition 3.5 with the aid of almost conservation law of frequency localized mass just as in [6], [31] and [35]. By the scaling (1.4), we may take $N_{min} = 1$.

7.1 The setup and contradiction argument

Since $N(t) \in 2^{\mathbb{Z}}$, there exists $t_{min} \in I_0$ such that $N(t_{min}) = N_{min} = 1$.

At time $t = t_{min}$, we have a considerable amount of mass at medium frequencies:

$$\|P_{C(\eta_0) < \cdot < C(\eta_0)} u(t_{min})\|_{L_x^2} \gtrsim c(\eta_0). \quad (7.1)$$

However, by the Bernstein estimate, there is not much mass at frequencies higher than $C(\eta_0)$

$$\|P_{> C(\eta_0)} u(t_{min})\|_{L_x^2} \lesssim c(\eta_0).$$

Let's assume for a contradiction that there exists $t_{evac} \in I_0$ such that $N(t_{evac}) \gg C(\eta_4)$. By time reversal symmetry, we may assume $t_{min} < t_{evac}$. If $C(\eta_4)$ is sufficiently large, we then see from Corollary 3.1 that energy has been almost entirely evacuated from low and medium frequencies at time t_{evac} :

$$\|P_{< \frac{1}{\eta_4}} u(t_{evac})\|_{\dot{H}_x^1} \leq \eta_4. \quad (7.2)$$

We define

$$u_{lo}(t) = P_{< \eta_3^{10n}} u(t), \quad u_{hi}(t) = P_{\geq \eta_3^{10n}} u(t).$$

Then by (7.1),

$$\|u_{hi}(t_{min})\|_{L_x^2} \geq \eta_1. \quad (7.3)$$

Suppose we could show that a big portion of the mass sticks around until time t_{evac} , i.e.,

$$\|u_{hi}(t_{evac})\|_{L_x^2} \geq \frac{1}{2} \eta_1. \quad (7.4)$$

Since we have by the Bernstein estimate

$$\|P_{> C(\eta_1)} u_{hi}(t_{evac})\|_{L_x^2} \leq c(\eta_1),$$

then the triangle inequality would imply

$$\|P_{\leq C(\eta_1)} u_{hi}(t_{evac})\|_{L_x^2} \geq \frac{1}{4} \eta_1.$$

Another application of the Bernstein estimate would give

$$\|P_{\leq C(\eta_1)} u(t_{evac})\|_{\dot{H}_x^1} \gtrsim c(\eta_1, \eta_3),$$

which would contradict (7.2) if η_4 were chosen sufficiently small.

It therefore remains to show (7.4). In order to prove it we assume that there exists a time t_* such that $t_{min} \leq t_* \leq t_{evac}$ and

$$\inf_{t_{min} \leq t \leq t_*} \|u_{hi}(t)\|_{L_x^2} \geq \frac{1}{2} \eta_1. \quad (7.5)$$

We will show that this can be bootstrapped to

$$\inf_{t_{min} \leq t \leq t_*} \|u_{hi}(t)\|_{L_x^2} \geq \frac{3}{4} \eta_1. \quad (7.6)$$

Hence, $\{t_* \in [t_{min}, t_{evac}] : (7.5) \text{ holds}\}$ is both open and closed in $[t_{min}, t_{evac}]$ and (7.4) holds.

In order to show that (7.5) implies (7.6), we will treat the L_x^2 -norm of u_{hi} as an almost conserved quantity. Define

$$L(t) = \int_{\mathbb{R}^n} |u_{hi}(t, x)|^2 dx.$$

By (7.3), we have $L(t_{min}) \geq \eta_1^2$. Hence, by the Fundamental Theorem of Calculus it suffices to show that

$$\int_{t_{min}}^{t_*} |\partial_t L(t)| dt \leq \frac{1}{100} \eta_1^2.$$

Since

$$\partial_t L(t) = 2 \int_{\mathbb{R}^n} \{P_{hi}(|\nabla|^{-(n-4)}|u|^2 u), u_{hi}\}_m dx = 2 \int_{\mathbb{R}^n} \{P_{hi}(|\nabla|^{-(n-4)}|u|^2 u) - |\nabla|^{-(n-4)}|u_{hi}|^2 u_{hi}, u_{hi}\}_m dx,$$

we need to show

$$\int_{t_{min}}^{t_*} \left| \int_{\mathbb{R}^n} \{P_{hi}(|\nabla|^{-(n-4)}|u|^2 u) - |\nabla|^{-(n-4)}|u_{hi}|^2 u_{hi}, u_{hi}\}_m dx \right| dt \leq \frac{1}{100} \eta_1^2. \quad (7.7)$$

In order to prove (7.7), we need to control the various interactions between low, medium, and high frequencies. In the next section we will make some preliminary estimates that will make this goal possible.

7.2 Spacetime estimates for high, medium, and low frequencies

Remember that the frequency-localized interaction Morawetz inequality implies that for $N < c(\eta_2)N_{min}$,

$$\int_{t_{min}}^{t_{evac}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|P_{\geq N} u(t, y)|^2 |P_{\geq N} u(t, x)|^2}{|x - y|^3} dx dy dt \lesssim \eta_1 N^{-3}.$$

This estimate is useful for medium and high frequencies; however it is extremely bad for low frequencies since N^{-3} gets increasingly larger as $N \rightarrow 0$. We therefore need to develop better estimates in this case. Since $u_{\leq \eta_3}$ has extremely small energy at $t = t_{evac}$ (see (7.2)), we expect it to have small energy at all time in $[t_{min}, t_{evac}]$. Of course, there is energy leaking from the high frequencies to the low frequencies, but the interaction Morawetz estimate limits this leakage. Indeed, we have

Proposition 7.1. *under the assumptions above,*

$$\|P_{\leq N} u\|_{\dot{S}^1([t_{min}, t_{evac}] \times \mathbb{R}^n)} \lesssim \eta_4 + \eta_0 \eta_3^{-2} N^2,$$

for all $N \leq \eta_3$.

Proof: Consider the set

$$\Omega = \{t \in [t_{min}, t_{evac}] : \|P_{\leq N} u\|_{\dot{S}^1([t, t_{evac}] \times \mathbb{R}^n)} \leq C_0 \eta_4 + \eta_0 \eta_3^{-2} N^2\}$$

where C_0 is a large constant to be chosen later and not depending on any of the η 's.

Our goal is to show that $t_{min} \in \Omega$. First we can show that $t \in \Omega$ for t close to t_{evac} .

Now suppose that $t \in \Omega$. We will show that

$$\|P_{\leq N} u\|_{\dot{S}^1([t, t_{evac}] \times \mathbb{R}^n)} \leq \frac{1}{2} C_0 \eta_4 + \frac{1}{2} \eta_0 \eta_3^{-2} N^2 \quad (7.8)$$

holds for any $N \leq \eta_3$. thus, Ω is both open and closed in $[t_{min}, t_{evac}]$ and we have $t_{min} \in \Omega$ as desired.

Fixing $N \leq \eta_3$, the Strichartz estimate implies

$$\|P_{\leq N} u\|_{\dot{S}^1([t, t_{evac}] \times \mathbb{R}^n)} \lesssim \|P_{\leq N} u(t_{evac})\|_{\dot{H}_x^1} + \|\nabla P_{\leq N}(|\nabla|^{-(n-4)}|u|^2 u)\|_{L^{\frac{3}{2}} L^{\frac{6n}{3n+4}}}.$$

By (7.2), we have

$$\|P_{\leq N}u(t_{vac})\|_{\dot{H}_x^1} \lesssim \eta_4 \quad (7.9)$$

which is acceptable for (7.8) if C_0 is chosen sufficiently large.

To handle the nonlinearity, we decompose $u = u_{<\eta_4} + u_{\eta_4 \leq \cdot \leq \eta_3} + u_{>\eta_3}$ and use the triangle inequality to estimate

$$\|\nabla P_{\leq N}(|\nabla|^{-(n-4)}|u|^2u)\|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \lesssim \|\nabla P'_{\leq N}(|\nabla|^{-(n-4)}|u_{<\eta_4}|^2|u_{<\eta_4}|)\|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \quad (7.10)$$

$$+ \|\nabla P'_{\leq N}(|\nabla|^{-(n-4)}|u_{<\eta_4}|^2|u_{\eta_4 \leq \cdot \leq \eta_3}|)\|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \quad (7.11)$$

$$+ \|\nabla P'_{\leq N}(|\nabla|^{-(n-4)}|u_{<\eta_4}|^2|u_{>\eta_3}|)\|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \quad (7.12)$$

$$+ \|\nabla P'_{\leq N}(|\nabla|^{-(n-4)}|u_{\eta_4 \leq \cdot \leq \eta_3}|^2|u_{<\eta_4}|)\|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \quad (7.13)$$

$$+ \|\nabla P'_{\leq N}(|\nabla|^{-(n-4)}|u_{\eta_4 \leq \cdot \leq \eta_3}|^2|u_{\eta_4 \leq \cdot \leq \eta_3}|)\|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \quad (7.14)$$

$$+ \|\nabla P'_{\leq N}(|\nabla|^{-(n-4)}|u_{\eta_4 \leq \cdot \leq \eta_3}|^2|u_{>\eta_3}|)\|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \quad (7.15)$$

$$+ \|\nabla P'_{\leq N}(|\nabla|^{-(n-4)}|u_{>\eta_3}|^2|u|)\|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \quad (7.16)$$

Using the bootstrap hypothesis $t \in \Omega$, we have

$$\begin{aligned} \|u_{\eta_4 \leq \cdot \leq \eta_3}\|_{\dot{S}^0} &\lesssim \sum_{\eta_4 \leq M \leq \eta_3} \|P_M u\|_{\dot{S}^0} \lesssim \sum_{\eta_4 \leq M \leq \eta_3} M^{-1} \|\nabla P_M u\|_{\dot{S}^0} \\ &\lesssim \sum_{\eta_4 \leq M \leq \eta_3} M^{-1} \|P_M u\|_{\dot{S}^1} \lesssim \sum_{\eta_4 \leq M \leq \eta_3} M^{-1} (C_0 \eta_4 + \eta_0 \eta_3^{-2} M^2) \lesssim \eta_0 \eta_3^{-1}. \end{aligned} \quad (7.17)$$

We now turn to estimate the nonlinearity. Using the bootstrap hypothesis, $t \in \Omega$, we estimate

$$\begin{aligned} (7.10) &\lesssim \|\nabla P'_{\leq N}(|\nabla|^{-(n-4)}|u_{<\eta_4}|^2|u_{<\eta_4}|)\|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \\ &\lesssim \|\nabla u_{<\eta_4}\|_{L^3L^{\frac{6n}{3n-4}}} \|u_{<\eta_4}\|_{L^6L^{\frac{6n}{3n-8}}}^2 \lesssim \|u_{<\eta_4}\|_{\dot{S}^1}^3 \lesssim (C_0 \eta_4 + \eta_0 \eta_3^{-2} \eta_4^2)^3 \lesssim \eta_4, \end{aligned}$$

which again is acceptable for (7.8) provided C_0 is sufficiently large.

By the Bernstein estimate, (7.17), Corollary 3.2 and $t \in \Omega$, we obtain

$$\begin{aligned} (7.11) &\lesssim N \| |\nabla|^{-(n-4)} |u_{<\eta_4}|^2 |u_{\eta_4 \leq \cdot \leq \eta_3}| \|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \\ &\lesssim N \|u_{<\eta_4}\|_{L^6L^{\frac{6n}{3n-8}}}^2 \|u_{\eta_4 \leq \cdot \leq \eta_3}\|_{L^3L^{\frac{6n}{3n-4}}} \\ &\lesssim N (C_0 \eta_4 + \eta_0 \eta_3^{-2} \eta_4^2)^2 \eta_0 \eta_3^{-1} \lesssim \eta_4, \end{aligned}$$

and

$$\begin{aligned} (7.12) &\lesssim N \| |\nabla|^{-(n-4)} |u_{<\eta_4}|^2 |u_{>\eta_3}| \|_{L^{\frac{3}{2}}L^{\frac{6n}{3n+4}}} \\ &\lesssim N \|u_{<\eta_4}\|_{L^6L^{\frac{6n}{3n-8}}}^2 \|u_{>\eta_3}\|_{L^3L^{\frac{6n}{3n-4}}} \\ &\lesssim N (C_0 \eta_4 + \eta_0 \eta_3^{-2} \eta_4^2)^2 \eta_1^{\frac{1}{3}} \eta_3^{-1} \lesssim \eta_4, \end{aligned}$$

which again is acceptable for (7.8) provided C_0 is sufficiently large.

By the Bernstein estimate, (7.17), Corollary 3.2 and $t \in \Omega$, we have

$$\begin{aligned}
(7.13) &\lesssim N^2 \left\| |\nabla|^{-(n-4)} |u_{\eta_4 \leq \cdot \leq \eta_3}|^2 |u_{< \eta_4}| \right\|_{L^{\frac{3}{2}} L^{\frac{6n}{3n+10}}} \\
&\lesssim N^2 \left\| u_{< \eta_4} \right\|_{L^\infty L^{\frac{2n}{n-2}}} \left\| u_{\eta_4 \leq \cdot \leq \eta_3} \right\|_{L^3 L^{\frac{6n}{3n-4}}}^2 \\
&\lesssim N^2 (C_0 \eta_4 + \eta_0 \eta_3^{-2} \eta_4^2) (\eta_0 \eta_3^{-1})^2 \lesssim \eta_4, \\
(7.14) &\lesssim N^2 \left\| |\nabla|^{-(n-4)} |u_{\eta_4 \leq \cdot \leq \eta_3}|^2 u_{\eta_4 \leq \cdot \leq \eta_3} \right\|_{L^{\frac{3}{2}} L^{\frac{6n}{3n+10}}} \\
&\lesssim N^2 \left\| u_{\eta_4 \leq \cdot \leq \eta_3} \right\|_{L^\infty L^{\frac{2n}{n-2}}} \left\| u_{\eta_4 \leq \cdot \leq \eta_3} \right\|_{L^3 L^{\frac{6n}{3n-4}}}^2 \\
&\lesssim N^2 (\eta_0 \eta_3^{-1})^2 = \eta_0^2 \eta_3^{-2} N^2,
\end{aligned}$$

and

$$\begin{aligned}
(7.15) &\lesssim N^2 \left\| |\nabla|^{-(n-4)} |u_{\eta_4 \leq \cdot \leq \eta_3}|^2 u_{> \eta_3} \right\|_{L^{\frac{3}{2}} L^{\frac{6n}{3n+10}}} \\
&\lesssim N^2 \left\| u_{\eta_4 \leq \cdot \leq \eta_3} \right\|_{L^6 L^{\frac{6n}{3n-8}}} \left\| u_{\eta_4 \leq \cdot \leq \eta_3} \right\|_{L^6 L^{\frac{6n}{3n-2}}} \left\| u_{> \eta_3} \right\|_{L^3 L^{\frac{6n}{3n-4}}} \\
&\lesssim N^2 \eta_0 \eta_3^{-1} \eta_1^{\frac{1}{3}} \eta_3^{-\frac{1}{3}} = \eta_0 \eta_1^{\frac{1}{3}} \eta_3^{-2} N^2,
\end{aligned}$$

which again is acceptable for (7.8) provided C_0 is sufficiently large.

By the Bernstein estimate, (7.17), Corollary 3.2 and $t \in \Omega$, we have

$$\begin{aligned}
(7.16) &\lesssim N^2 \left\| |\nabla|^{-(n-4)} |u_{\geq \eta_3}|^2 u \right\|_{L^{\frac{3}{2}} L^{\frac{6n}{3n+10}}} \\
&\lesssim N^2 \left\| u \right\|_{L^\infty L^{\frac{2n}{n-2}}} \left\| u_{> \eta_3} \right\|_{L^3 L^{\frac{6n}{3n-4}}}^2 \lesssim N^2 (\eta_1^{\frac{1}{3}} \eta_3^{-1})^2 = \eta_1^{\frac{2}{3}} \eta_3^{-2} N^2
\end{aligned}$$

which again is acceptable for (7.8) provided C_0 is sufficiently large.

The proposition is complete.

7.3 Controlling the localized L^2 mass increment

We now have good enough control over low, medium, and high frequencies to prove (7.7). Writing

$$\begin{aligned}
P_{hi}(|\nabla|^{-(n-4)} |u|^2 u) - |\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi} &= P_{hi}(|\nabla|^{-(n-4)} |u|^2 u - |\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi} - |\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo}) \\
&\quad - P_{lo}(|\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi}) + P_{hi}(|\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo}).
\end{aligned}$$

Clearly, we only have to consider the following terms

$$\int_{t_{min}}^{t_*} \left| \int_{\mathbb{R}^n} \bar{u}_{hi} P_{hi}(|\nabla|^{-(n-4)} |u|^2 u - |\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi} - |\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo}) dx \right| dt. \quad (7.18)$$

$$\int_{t_{min}}^{t_*} \left| \int_{\mathbb{R}^n} \bar{u}_{hi} P_{lo}(|\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi}) dx \right| dt. \quad (7.19)$$

$$\int_{t_{min}}^{t_*} \left| \int_{\mathbb{R}^n} \bar{u}_{hi} P_{hi}(|\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo}) dx \right| dt. \quad (7.20)$$

• Case 1. Estimation of (7.18).

We move the self-adjoint operator P_{hi} onto \bar{u}_{hi} , and obtain

$$\begin{aligned}
&|\nabla|^{-(n-4)} |u|^2 u - |\nabla|^{-(n-4)} |u_{hi}|^2 u_{hi} - |\nabla|^{-(n-4)} |u_{lo}|^2 u_{lo} \\
&= |\nabla|^{-(n-4)} |u_{lo}|^2 u_{hi} + 2|\nabla|^{-(n-4)} \text{Re}(u_{lo} \bar{u}_{hi})(u_{lo} + u_{hi}) + |\nabla|^{-(n-4)} |u_{hi}|^2 u_{lo}.
\end{aligned}$$

We first consider the contribution of $|\nabla|^{-(n-4)}|u_{lo}|^2u_{hi}$. By Corollary 3.2 and Proposition 7.1, we have

$$\begin{aligned} \int_{t_{min}}^{t_*} \int_{\mathbb{R}^n} |P_{hi}u_{hi}| |\nabla|^{-(n-4)}|u_{lo}|^2|u_{hi}| dx dt &\lesssim \|P_{hi}u_{hi}\|_{L^3L^{\frac{6n}{3n-4}}} \|u_{hi}\|_{L^3L^{\frac{6n}{3n-4}}} \|u_{lo}\|_{L^6L^{\frac{6n}{3n-8}}}^2 \\ &\lesssim (\eta_1(\eta_3^{10n})^{-3})^{\frac{2}{3}} (\eta_4 + (\eta_3^{-1}\eta_3^{10n})^2)^2 \\ &\lesssim \eta_1^{\frac{2}{3}} \eta_3^{20n-4} \ll \eta_1^2. \end{aligned}$$

We now turn towards the contribution of $|\nabla|^{-(n-4)}|u_{hi}|^2u_{lo}$. We decompose $u_{hi} = u_{\eta_3^{10n} \leq \cdot \leq \eta_3} + u_{>\eta_3}$ and obtain by the Bernstein estimate, Corollary 3.2 and Proposition 7.1,

$$\begin{aligned} \|u_{>\eta_3}\|_{L^3L^{\frac{6n}{3n-4}}} &\lesssim \eta_1^{\frac{1}{3}} \eta_3^{-1}, \\ \|u_{\eta_3^{10n} \leq \cdot \leq \eta_3}\|_{L^3L^{\frac{6n}{3n-4}}} &\lesssim \sum_{\eta_3^{10n} \leq N \leq \eta_3} \|u_N\|_{L^3L^{\frac{6n}{3n-4}}} \lesssim \sum_{\eta_3^{10n} \leq N \leq \eta_3} N^{-1} \|\nabla u_N\|_{L^3L^{\frac{6n}{3n-4}}} \\ &\lesssim \sum_{\eta_3^{10n} \leq N \leq \eta_3} N^{-1} (\eta_4 + \eta_3^{-2} N^2) \lesssim \eta_3^{-2} \eta_3 = \eta_3^{-1}, \\ \|u_{lo}\|_{L^\infty L^{\frac{2n}{n-4}}} &\lesssim \eta_3^{10n} \|u_{lo}\|_{L^\infty L^{\frac{2n}{n-2}}} \lesssim \eta_3^{10n} (\eta_4 + \eta_3^{-2} (\eta_3^{10n})^2) = \eta_3^{30n-2}, \end{aligned}$$

then

$$\begin{aligned} \int_{t_{min}}^{t_*} \int_{\mathbb{R}^n} |P_{hi}u_{hi}| |\nabla|^{-(n-4)}|u_{hi}|^2|u_{lo}| dx dt &\lesssim \sum_{j=0}^3 \|u_{\eta_3^{10n} \leq \cdot \leq \eta_3}\|_{L^3L^{\frac{6n}{3n-4}}}^{3-j} \|u_{>\eta_3}\|_{L^3L^{\frac{6n}{3n-4}}}^j \|u_{lo}\|_{L^\infty L^{\frac{2n}{n-4}}} \\ &\lesssim \sum_{j=0}^3 (\eta_1^{\frac{1}{3}} \eta_3^{-1})^j (\eta_3^{-1})^{3-j} \eta_3^{30n-2} \lesssim \eta_3^{30n-5} \ll \eta_1^2. \end{aligned}$$

Now we consider the contribution of $2|\nabla|^{-(n-4)}\text{Re}(u_{lo}\bar{u}_{hi})(u_{lo} + u_{hi})$. Similarly, we have

$$\begin{aligned} \int_{t_{min}}^{t_*} \int_{\mathbb{R}^n} |P_{hi}u_{hi}| |\nabla|^{-(n-4)}(u_{lo}\bar{u}_{hi})(u_{lo} + u_{hi}) dt &\lesssim \|P_{hi}u_{hi}\|_{L^3L^{\frac{6n}{3n-4}}} \|u_{hi}\|_{L^3L^{\frac{6n}{3n-4}}} \|u_{lo}\|_{L^6L^{\frac{6n}{3n-8}}}^2 \\ &+ \sum_{j=0}^3 \|u_{\eta_3^{10n} \leq \cdot \leq \eta_3}\|_{L^3L^{\frac{6n}{3n-4}}}^j \|u_{>\eta_3}\|_{L^3L^{\frac{6n}{3n-4}}}^{3-j} \|u_{lo}\|_{L^\infty L^{\frac{2n}{n-4}}} \\ &\lesssim \eta_1^{\frac{2}{3}} \eta_3^{20n-4} + \eta_3^{30n-5} \ll \eta_1^2. \end{aligned}$$

Therefore

$$(7.18) \ll \eta_1^2.$$

• **Case 2. Estimation of (7.19).**

Moving the projection P_{lo} onto \bar{u}_{hi} and writing $P_{lo}u_{hi} = P_{hi}u_{lo}$, and get

$$\begin{aligned} (7.19) &= \int_{t_{min}}^{t_*} \left| \int_{\mathbb{R}^n} P_{hi}\bar{u}_{lo}(|\nabla|^{-(n-4)}|u_{hi}|^2u_{hi}) dx \right| dt \\ &\lesssim \sum_{j=0}^3 \|u_{\eta_3^{10n} \leq \cdot \leq \eta_3}\|_{L^3L^{\frac{6n}{3n-4}}}^j \|u_{>\eta_3}\|_{L^3L^{\frac{6n}{3n-4}}}^{3-j} \|P_{hi}u_{lo}\|_{L^\infty L^{\frac{2n}{n-4}}} \\ &\lesssim \sum_{j=0}^3 \|u_{\eta_3^{10n} \leq \cdot \leq \eta_3}\|_{L^3L^{\frac{6n}{3n-4}}}^j \|u_{>\eta_3}\|_{L^3L^{\frac{6n}{3n-4}}}^{3-j} \|u_{lo}\|_{L^\infty L^{\frac{2n}{n-4}}} \lesssim \eta_3^{30n-5} \ll \eta_1^2. \end{aligned}$$

• **Case 3. Estimation of (7.20).**

By Bernstein estimate, Corollary 3.2 and Proposition 7.1, we have

$$\begin{aligned}
(7.20) &\lesssim \|u_{hi}\|_{L^3 L^{\frac{6n}{3n-4}}} \|P_{hi}(|\nabla|^{-(n-4)}|u_{lo}|^2 u_{lo})\|_{L^{\frac{3}{2}} L^{\frac{6n}{3n+4}}} \\
&\lesssim \|u_{hi}\|_{L^3 L^{\frac{6n}{3n-4}}} \|\nabla P_{hi}(|\nabla|^{-(n-4)}|u_{lo}|^2 u_{lo})\|_{L^{\frac{3}{2}} L^{\frac{6n}{3n+4}}} \\
&\lesssim \|u_{hi}\|_{L^3 L^{\frac{6n}{3n-4}}} \|\nabla u_{lo}\|_{L^3 L^{\frac{6n}{3n-4}}} \|u_{lo}\|_{L^6 L^{\frac{6n}{3n-8}}}^2 \\
&\lesssim \eta_1^{\frac{1}{3}} (\eta_3^{10n})^{-1} (\eta_4 + (\eta_3^{-1} \eta_3^{10n})^2)^3 = \eta_1^{\frac{1}{3}} \eta_3^{50n-6} \ll \eta_1^2.
\end{aligned}$$

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